ANALYTICAL RESOLUTION OF THE CAUCHY PROBLEM OF A DIFFUSION EQUATION WITH A FRACTIONAL TIME DERIVATIVE.

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Abstract.

In this paper, we will present a method of analytical resolution of a fractional time derivative diffusion equation with an initial condition. Spectral theory of the operators, Hilbert space, some properties of Mittag-Leffler function, Fourier transformation of the generalized functions are main tools for this resolution.

Key words : *Fractional derivative, Generalized functions, Mittag-Leffler function.*

1 INTRODUCTION

The concept of the fractional derivative is a subject almost old than classical calculation that we know today. However this theory can be considered as new subject too, since a little more than thirty years. Recently, there has been considerable development in the resolution of fractional differential equations (see examples in [1], [2], [3])

The equation to solve is :

$$ {}_{RL}D_{0^+}^{\alpha}(u(t,x)-u(0,x))-\Delta_x(t,x)=f(t,x) $$

Where :

- $0 \prec \alpha \prec 1$
- u is the unknown, $(t,x)\in[0,T[X\Omega$, $\Omega=\prod_{p=1}^{N}[0,L_P]\subset IR^N$
- u(0,x)=$u_0(x)$ for any x element of Ω
- $u(t,x)=0$ in $]0,T[\,X\partial\Omega$
- f is a given function
- ${}_{RL}D_{O^+}^{\alpha}(u(t,x)-u(0,x))=\dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dt}\int_{0^+}^{t}(t-\tau)^{-\alpha}u(t,x)-u(x,0)d\tau$ (Riemann-Liouville derivative of $u(t,x)-u(0,x)$ si $0\prec\alpha\prec 1$)
- For $t\in[0,T]$, $u(t,x)\in L^2(\Omega)$ and f(t, x)

Its resolution requires a spectral decomposition of the Laplacian operator ($-\Delta_x$) where $\Omega=\prod_{p=1}^{N}[0,L_p]\subset IR^N$, eigenvalues are given in their explicit forms.

The eigenvectors obtained from this decomposition form a Hilbert base of $L^2(\Omega)$.

Relationships between the coordinates of u and f in this base form linear fractional differential equations including Fourier transform, fractional derivation of a distribution, some properties of products of convolutions of generalized functions are main means to find the solution and the function of Mittag-Leffler simplifies the presentation of the solution of this differential equation.

This paper will be presented as follows:

Part will be devoted to reminders about the different tools needed for the resolution

Another part for the spectral decomposition of the Laplacian operator $(-\Delta_x)$ where $\Omega = \prod_{p=1}^{N}\left[0, L_p\right] \subset IR^N$ with Dirichlet conditions.

A last part, before the conclusion, is the resolution of the equation.

2 PRELIMINARY KWNOLEDGE

In this part, we will introduce definitions, properties and notations necessary for this note

- The fractional differential operator of Riemann-Liouville:
  ${}_{RL}D^{\alpha}_{0^+}$ as ${}_{RL}D^{\alpha}_{0^+}u(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\tau)^{-\alpha}u(\tau)d\tau$ si $0 \prec \alpha \prec 1$ See [1],[2],[3],[4]
  u is a continuous and differentiable numeric function in $]0,T[$
- The differential operator of Caputo :
  ${}_{c}D^{\alpha}_{0^+}$ as ${}_{c}D^{\alpha}_{0^+} = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{d}{d\tau}u(\tau)d\tau = {}_{RL}D^{\alpha}_{0^+}(u(t)-u(0^+))$
- If u is causal then ${}_{RL}D^{\alpha}_{0^+} = {}_{c}D^{\alpha}_{0+} = {}_{-\infty}D^{\alpha}_{0^+} = D^{\alpha}$ voir [1], [2] (1)
- Fourier transformation of $D^{\alpha}u$ is $\mathfrak{F}[D^{\alpha}u](\omega) = (i\omega)^{\alpha}\ \mathfrak{F}[u]$ See [1], [2] (2)
- In the sense of temperate distributions, we define $\mathfrak{F}\ [D^{\alpha}U] = (i\omega)^{\alpha}\ \mathfrak{F}\ [U]$ See [2]
- For a causal function u, $D^{\alpha}u(t) = 0$ if, and only if, $u(t) = \frac{\Gamma(1)}{\Gamma(\alpha)}ct^{\alpha-1}$ (3)
  $0 \prec \alpha \prec 1$ and $t \geq 0$, $c \in IR$ See [2]
- $Y_{\lambda}(t) = \frac{t^{\lambda-1}}{\Gamma(\lambda)}Y(t), \lambda \in IR - \{0,-1,-2,......\}; Y_1 = Y; Y_0 = \delta_0$ [2]
- If $0 \prec \alpha \prec 1$ and T a compact support distribution, then $D^{\alpha}T = DY_{1-\alpha} * T$ [2] (4)

- For $\alpha, \beta \succ 0$, Mittag-Leffler function is defined as

$$E_{\alpha,\beta} = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)}\ , \qquad z \in C \tag{5}$$

- Let H be a separable Hilbert space and (e_n) a countable family orthonormed of H . The following assertions are equivalent :
  - (e_n) is a a Hilbertian base
  - For any $x \in H$, $x = \sum_{n \in IN}\langle x, e_n\rangle e_n$ (6)

- For any $x \in H$, $\|x\|^2 = \sum_{n \in IN} \left|\langle x, e_n \rangle\right|^2$ (7)
- $\{e_n \mid n \in IN\}^{\perp} = \{0\}$ (8)

- Let Ω be an open bounded class C^1 of IR^d. Then there exists an increasing sequence $(\lambda_k)_{k \geq 1}$ of real positives that goes to infinity and a Hilbertian base $(u_k)_{k \geq 1}$ of $L^2(\Omega)$ such $u_k \in H_0^1(\Omega)$ and $-\Delta u_k = \lambda_k u_k$ [18] [19]

3 – SPECTRAL DECOMPOSITION OF THE LAPLACIAN OPERATOR $-\Delta_x$ OF INITIAL PROBLEM

- Consider the case of $-\Delta_x$ for $\Omega = \left]0,1\right[$ and with Dirichlet conditions $u_k(x) = \sqrt{2}\sin(k\pi x)$ $k \in IN^*$ the eigenfunctions of $-\Delta_x$ and $\lambda_k = k^2\pi^2$ the eigenvalues corresponding to u_k See [12],[18]
- For $\Omega = \prod_{p=1}^{N} \left]0, L_p\right[\subset IR^N$. Let introduce $u_{p,k}$ function in $\left[0, L_p\right]$ with values in IR, defined by $u_{p,k}(x_p) = u_k\left(\frac{x_p}{L_p}\right)$ and for any $k = (k_1, k_2, ..., k_N) \in IN^N$ and for any $x = (x_1, ..., x_N) \in IR^N$, we set $v_k(x) = \prod_{p=1}^{N} u_{p,k_p}(x_p)$ (9)

$$-\Delta_x v_k(x) = -\Delta_x\left[\prod_{p=1}^{N} u_{p,k_p}(x_p)\right] = -\left(\sum_{p=1}^{N} \frac{\partial^2}{\partial x_p^2}\right)\prod_{p=1}^{N} u_{p,k_p}(x_p)$$

$$= \sum_{p=1}^{N}\left(\frac{k_p\pi}{L_p}\right)^2 \left(\prod_{p=1}^{N} u_{p,k_p}(x_p)\right) = \left[\sum_{p=1}^{N}\left(\frac{k_p\pi}{L_p}\right)^2\right] v_k(x)$$

Thus $\lambda_k = \sum_{p=1}^{N}\left(\frac{k_p\pi}{L_p}\right)^2, k = (k_1, k_2, ..., k_P)$ is the eigenvalue corresponding to v_k (10)

- Show that $(v_k)_{k \in IN^N}$ is a Hilbertian base of $L^2(\Omega)$

Just show that :

If $\langle v_k, w \rangle = 0$ for any $k \in IN^N$ then $w = 0$ and we can conclude that (v_k) is a Hilbertian base becasuse $L^2(\Omega)$ is a separable Hilbertian base according to (8)

Let's proceed by recurrence on the dimension N

This result is true for N=1 See [12],[13]

Suppose the result is established for Ω of dimension N-1 and we introduce the function $\tilde{\omega} \in L^2 \left]0, L_N\right[$ defined by $\tilde{\omega}(x_N) = \int_{\tilde{\Omega}} (w(x)) \prod_{p=1}^{N-1} u_{p,k_p}(x_p)\, d\tilde{x}$

where $\tilde{\Omega} = \prod_{p=1}^{N-1} \left]0, L_p\right[$ and $\tilde{x} = (x_1, ..., x_{N-1})$

By hypothesis $\langle v_k, w \rangle = 0$ for any $k \in IN^N$ so $\int_0^{L_N} \tilde{\omega}(x_N) u_{N,k}(x_N)\, dx_N = 0, k \in IN^*$

As $\left(u_{N,k}\right)_{k\in IN}$ is a base of $L^2([0,L_N])$, we conclude that $\omega(x_N)=0$ a.e.

Thus $w_{x_N}(x)=w\left(x,x_N\right)\in L^2(\Omega)$ such as $\int_{\Omega} w_{x_N}(x)\prod_{p=1}^{N-1}u_{k_p}\left(x_p\right)d\,x=0$

By recurrence hypothesis $w_{x_N}=0$ (what to show)

Then $L^2(\Omega)=\underset{k\in IN^N}{\oplus}E_k$, E_k is the generated space $\left(v_k\right)_{k\in IN^N}$, E_k vector spare generated by (v_k) See [13]

4 - RESOLUTION OF THE INITIAL EQUATION

$$ {}_{RL}D_{0^+}^{\alpha}\left[u(t,x)-u(0,x\right]-\Delta_x\left(t,x\right)=f(t,x)\quad (t,x)\in\left]0,T\right[X\Omega \qquad (11)$$

For one $t\in\left]0,T\right[$, $u(t,x)=\sum_{k\in IN^N}\langle u(t,x),v_k(x)\rangle v_k\left(x\right)=\sum_{k\in IN^N}c_k\left(t\right)v_k\left(t\right)$ according to (6)

$$u(0,x)=u_0=\sum_{k\in IN^N}c_k\left(0\right)v_k(x)\quad f(x,t)=\sum_{k\in IN^N}\beta_k(t)v_k(x)$$

et $-\Delta_x u(t,x)=\sum_{k\in IN^N}c_k(t)\lambda_k v_k(x)$ then

$$ {}_{RL}D_{0^+}^{\alpha}[\sum_{k\in IN^N}c_k\left(t\right)v_k\left(t\right)-c_k(0)v_k(x)]+\sum_{k\in IN^N}c_k\left(t\right)\lambda_k v_k\left(x\right)=\sum_{k\in IN^N}\beta_k\left(t\right)v_k\left(x\right)$$

Thus ${}_{RL}D_{0^+}^{\alpha}[c_k(t)-c_k(0)]+\lambda_k c_k(t)=\beta_k(t)\quad 0\prec t\prec T$

a- **Resolution of** ${}_{RL}D_{0^+}^{\alpha}(c(t)-c(0))+\lambda_k c(t)=\beta(t)\qquad t\in]0,T[$ or

$$ {}_{RL}D_{0^+}^{\alpha}(c(t))+\lambda_k c(t)=\beta(t)+D^{\alpha}c(0) \qquad (12)$$

Step 1 :

Transformation of the equation in the sense of generalized functions

Let U be a generalized function such as U(t)= c(t) if $t\in]0,T[$, U=0 if t<0 and t>T

We can write U(t)= c(t)[Y(t)- Y(t-T)]. We note $Y_T(t)=Y(t-T)$

$$D^{\alpha}U+\lambda_k U=\frac{d}{dt}(Y_{1-\alpha}*U)+\lambda_k U=Y_{1-\alpha}*\frac{d}{dt}[c(t)(Y(t)-Y_T(t))]+\lambda_k c(t)(Y(t)-Y_T(t))$$

(13)

$$=Y_{1-\alpha}*\frac{dc(t)}{dt}[Y(t)-Y_T(t)]+Y_{1-\alpha}*c(t)(\delta_0-\delta_T)+\lambda_k c(t)(Y(t)-Y_T(t))$$

$$=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-\xi)^{-\alpha}\frac{dc(\xi)}{d\xi}((Y-Y_T)(\xi))d\xi+Y_{1-\alpha}*c(0^+)\delta_0(t)-Y_{1-\alpha}*c(T)\delta_T(t)+\lambda_k c(t)(Y(t)-Y_T(t))$$

If 0<t<T

$$D^{\alpha}U+\lambda_k U=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-\xi)^{-\alpha}\frac{dc(\xi)}{d\xi}d\xi+\lambda_k c(t) \qquad (14)$$

$$= {}_{c}D_{0^+}^{\alpha}u(t)+{}_{c}D_{0^+}^{\alpha}c_k(t)+\lambda_k c(t)$$

$$= {}_{RL}D^{\alpha}_{0^+}[c_k(t)-c_k(0^+)]+\lambda_k c_k(t) = {}_{RL}D^{\alpha}_{0^+}[c_k(t)-c_k(0)]+{}_{RL}D^{\alpha}_{0^+}[c_k(0)-c_k(0^+)]+\lambda_k c_k(t)$$
$$=\beta_k(t)+{}_{RL}D^{\alpha}_{0^+}[c_k(0)-c_k(0^+)]$$

Then $D^{\alpha}U+\lambda_k U=\beta_k(t)+{}_{RL}D^{\alpha}_{0^+}[c_k(0)-c_k(0^+)]$ (15)

Step 2

Resolution of $D^{\alpha}V+\lambda_k V=\delta$ (16)

For any temperate distribution (Compact support generalized functions are temperates), we have :

$$\mathfrak{F}\,(D^{\alpha}V+\lambda_k V)=(i\omega)^{\alpha}\,\mathfrak{F}(V)+\lambda_k\,\mathfrak{F}\,(V)$$

Thus $\mathfrak{F}\,(V)=\dfrac{1}{(i\omega)^{\alpha}+\lambda_k}$

Search of the inverse by the Fourier transform of $\dfrac{1}{(i\omega)^{\alpha}+\lambda_k}$

$$\frac{1}{(i\omega)^{\alpha}+\lambda_k}=\frac{1}{\lambda_k}\left[\frac{1}{\frac{(i\omega)^{\alpha}}{\lambda_k}+1}\right]=\frac{1}{\lambda_k}\frac{1}{\frac{1}{\lambda_k(i\omega)^{-\alpha}}+1}=\frac{1}{\lambda_k}\frac{\lambda_k(i\omega)^{-\alpha}}{1+\lambda_k(i\omega)^{-\alpha}}=\frac{(i\omega)^{-\alpha}}{1+\lambda_k(i\omega)^{-\alpha}}$$

$$=(i\omega)^{-\alpha}\frac{1}{1-(-\lambda_k)(i\omega)^{-\alpha}}=(i\omega)^{-\alpha}\sum_{p=0}^{\infty}(-\lambda_k)^{p}[(i\omega)^{-\alpha}]^{p}\ ,\quad\left|\lambda_k\omega^{-\alpha}\right|\prec 1$$

$$=\sum_{p=0}^{\infty}(-\lambda_k)^{p}(i\omega)^{\alpha p-\alpha}=\sum_{p=0}^{\infty}\frac{(-\lambda_k)}{(i\omega)^{\alpha p+\alpha}}\frac{\Gamma(\alpha p+\alpha)}{\Gamma(\alpha p+\alpha)}=\sum_{p=0}^{\infty}\frac{(-\lambda_k)}{\Gamma(\alpha p+\alpha)}\int_0^{\infty}\frac{u^{\alpha p+\alpha-1}e^{-u}}{(i\omega)^{\alpha p+\alpha}}du$$

$$=\sum_{p=0}^{\infty}\frac{(-\lambda_k)^{p}}{\Gamma(\alpha p+\alpha)}\int_0^{\infty}\left(\frac{u}{i\omega}\right)^{\alpha p+\alpha-1}e^{-u}\frac{du}{i\omega}=\sum_{p=0}^{\infty}\frac{(-\lambda)^{p}}{\Gamma(\alpha p+\alpha)}\int_0^{\infty}t^{\alpha p+\alpha-1}e^{-i\omega t}dt\qquad\left(t=\frac{u}{i\omega}\right)$$

$$\int_0^{\infty}[\sum_{p=0}^{\infty}\frac{(-\lambda_k)^{p}}{\Gamma(\alpha p+\alpha)}t^{\alpha p+\alpha-1}]e^{-i\omega t}dt=\int_0^{\infty}[t^{\alpha-1}\sum_{p=0}^{\infty}\frac{(-\lambda_k)^{p}}{\Gamma(\alpha p+\alpha)}t^{\alpha p}]e^{-i\omega t}dt$$

$$=\mathfrak{F}[t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})Y(t)](\omega)$$

Then $V(t)=t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k t^{\alpha})Y(t)$ (17)

Hence the basic solution of

$$D^{\alpha}U+U=[\beta(t)+Y_{1-\alpha}*c(0^+)\delta_0(t)-{}_{RL}D^{\alpha}_{0^+}c(0^+)+{}_{RL}D^{\alpha}_{0^+}c(0)](Y(t)-Y_T(t)]$$

is $U_0(t)=t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k t^{\alpha})[Y(t)-Y_T(t)]$ (18)

Uniqueness of the basic solution:

Suppose there is a solution U_1 , then $W=U_1-U_0$ satisfies the equation $D^{\alpha}W+\lambda_k W=0$ thus $\mathfrak{F}$ $F(D^{\alpha}W+\lambda W)=[(i\omega)^{\alpha}+\lambda_k]\,\mathfrak{F}(W)=0$ so $[(-i\omega)^{\alpha}+\lambda_k][(i\omega)^{\alpha}+\lambda_k]\,\mathfrak{F}\,(W)=0$

or $[\omega^{2\alpha}+\lambda_k{}^2]\,\mathfrak{F}$ (W)=0

$\lambda_k \succ 0$ then $\mathfrak{F}\,(W)=0$ thus W=0

Step 3 : Solution of the equation

a. The solution of the generalized function equation is

$$U=U_0*[Y_{1-\alpha}*\frac{dc(t)}{dt}[Y(t)-Y_T(t)]+Y_{1-\alpha}*c(t)(\delta_0-\delta_T)+\lambda_k c(t)(Y(t)-Y_T(t))]$$

$U=U_0*[\beta(t)-{}_{RL}D^{\alpha}_{0^+}c_k(0^+)+{}_{RL}D^{\alpha}_{0^+}c_k(0\ )]$ when 0<t<T

Thus $c_k(t)=t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t\alpha)*[\beta_k(t)-{}_{RL}D^{\alpha}_{0^+}c_k(0^+)+{}_{RL}D^{\alpha}_{0^+}c_k(0)]$ (19)

$$c_k(t)=t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k t\alpha)*[{}_{RL}D^{\alpha}_{0^+}c_k(0)-{}_{RL}D^{\alpha}_{0^+}c_k(0^+)]+\int_0^t(t-\xi)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-\xi)^{-\alpha})\beta_k(\xi)d\xi$$

0<t<T

b. Solution of the initial equation

The solution of the equation is $u(t,x)=\sum_{k\in IN^N}\langle u(t,x),v_k(x)\rangle v_k(x)=\sum_{k\in IN^N}c_k(t)v_k(t)$ where 0<t<T,

So just replace $c_k(t)$ with its expression.

5 CONCLUSION

In this paper, a method for solving a fractional differential equation from generalized functions has been provided. We have obtained a solution that reveals some characteristic points, but our obstacle is the fractional derivation requires that the generalized function must be compact support because of the convolution products.

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