# AN ANALYZING OF SECOND ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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## ABSTRACT

*The first value problems of ordinary differential equation systems are dealt with in this study. Our emphasis here is on physically related nonlininary phenomena and properties. The solution to a differential equation may not be so important if it does not occur or only under extraordinary circumstances in the physical model described by the system. Equilibrium solutions that are in line with physical device configurations that don't change occur even in everyday conditions, if they are stable. Throughout practice, an unstable equilibrium is not established, as minor disturbances in the system or its physical environment instantly break the system away from equilibrium.*

***Keywords:*** *Second Order, Differential Equation, Nonlinear, Ordinary*

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## 1. INTRODUCTION

Few nonlinear problems can be directly solved, so that the solution is typically approximated by a numerical method. The final part of the Chapter is devoted to the basic methods for initial value problems, from the simple Euler approach to the very famous fourth order Runge – Kutta method. Numerical systems do not always produce exact results, though, and we discuss briefly the form of rigid differential equations for numerical analysts which pose a major challenge. For any important theoretical understanding, it is not clear if numerical solutions (also those given by well-known packages) are to be used without a simple comprehension of the existence of solutions, equilibrium points and stability properties. This also helps to create a collection of non-linear concerns, of which one already knows one or more explicit theoretical solutions, by evaluating a computational scheme. Additional experiments and theoretical findings may be based on first integrals or Lyapunov functions in general. While we only have space to touch briefly on these subjects, we hope this will give the reader a greater interest in exploring this subject. You may usefully search references [2, 9, 13, 15, 17].

## 2. FIRST ORDER SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS.

Let us begin with bringing into discrete dynamics the fundamental object of analysis, the initial value problem of ordinary differential equations in a first order system. Most practical applications lead to higher order systems of ordinary differential equations, but a quick reformulation transforms them into first order equivalent systems. And, by restricting our focus to first order instances, we do not lose any generality. In addition, numerical solutions systems are entirely based on their reformulation as first order systems for problems in higher order initial values.

**Scalar Ordinary Differential Equations**

As always, when confronted with a new problem, it is essential to fully understand the simplest case first. Thus, we begin with a single scalar, first order ordinary differential equation

$$\frac{du}{dt} = F(t, u). \quad (1)$$

The independent t variable represents time in many applications, and the unknown u(t) function represents some complex physical quantity. All amounts are presumed to be actual in this chapter. Under the correct conditions on the left hand side (as formalized in the following section), the solution u(t) is defined by value at one time only, (Results on composing ordinary differential equations can be found on [14]);

$$u(t_0) = u_0. \quad (2)$$

The combination (.1–2) is considered an initial value problem and our goal is to evaluate theoretical and numerical solutions. When the right hand doesn't depend directly on the time variable, a differential equation is considered autonomous:

$$\frac{du}{dt} = F(u). \quad (3)$$

All autonomous scalar equations can be solved by direct integration. We divide both sides by F(u), whereby

$$\frac{1}{F(u)}\,\frac{du}{dt} = 1,$$

and then integrate with respect to t; the result is

$$\int \frac{1}{F(u)}\,\frac{du}{dt}\,dt = \int dt = t + k,$$

where k is the constant of integration. The left hand integral can be evaluated by the change of variables that replaces t by u, whereby du = (du/dt) dt, and so

$$\int \frac{1}{F(u)}\,\frac{du}{dt}\,dt = \int \frac{du}{F(u)} = G(u)$$

where G(u) indicates a convenient anti-derivative† of the function 1/F(u). Thus, the solution can be written in implicit form

$$G(u) = t + k. \quad (4)$$

If we are able to solve the implicit equation (4), we may thereby obtain the explicit solution

$$u(t) = H(t + k) \quad (5)$$

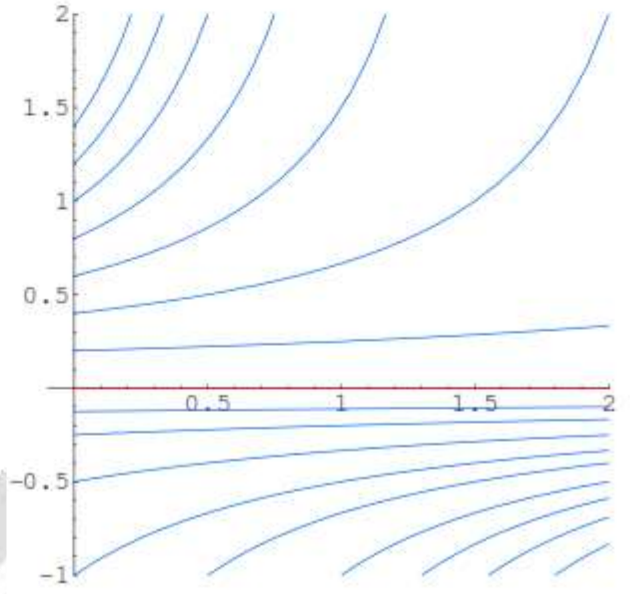


**Figure 1. Solutions to** $\dot{u} = u^2.$

In terms of the inverse function H = G−1 . Finally, to satisfy the initial condition (2), we set t = t0 in the implicit solution formula (4), whereby G(u0 ) = t0 + k. Therefore, the solution to our initial value problem is

$$G(u) - G(u_0) = t - t_0, \qquad \text{or, explicitly,} \qquad u(t) = H\big(t - t_0 + G(u_0)\big). \tag{6}$$

Remark: A more direct version of this solution technique is to rewrite the differential equation (3) in the "separated form

$$\frac{du}{F(u)} = dt,$$

in which all terms involving u, including its differential du, are collected on the left hand side of the equation, while all terms involving t and its differential are placed on the right, and then formally integrate both sides, leading to the same implicit solution formula:

$$G(u) = \int \frac{du}{F(u)} = \int dt = t + k. \tag{7}$$

Before completing our analysis of this solution method, let us run through a couple of elementary examples.

**Example 1.** Consider the autonomous initial value problem

$$\frac{du}{dt} = u^2, \qquad u(t_0) = u_0. \tag{8}$$

To solve the differential equation, we rewrite it in the separated for

$$\frac{du}{u^2} = dt, \quad \text{and then integrate both sides:} \quad -\frac{1}{u} = \int \frac{du}{u^2} = t + k.$$

Solving the resulting algebraic equation for u, we deduce the solution formula

$$u = -\frac{1}{t+k}. \quad (9)$$

To specify the integration constant k, we evaluate u at the initial time t0 ; this implies

$$u_0 = -\frac{1}{t_0+k}, \qquad \text{so that} \qquad k = -\frac{1}{u_0} - t_0.$$

Therefore, the solution to the initial value problem is

$$u = \frac{u_0}{1 - u_0(t - t_0)}. \quad (10)$$

Figure 1 shows the graphs of some typical solutions.

When the crític value t = t0 + 1/u0 is approached from below, "blows up' solution meaning u(t) = t = t + 1/u0 The blow-up time t oscillates according to the initial data — the bigger u0 > 0, the more easily the solution goes to infinity. If the first data is negative, u0 < 0, the solution for all t > t0 is well known, but in the past has a particular characteristic: t = t0 + 1 / u0 < t0. In both positive and negative times the only solution remains is the constant solution u(t) = 0, which corresponds to the initial condition u0 = 0.

Generally speaking, constant equilibrium solutions, also known as its fixed points, play an important role to an independent, ordinary differential equation. If u(t) — in other words, the differential equation ( 3) means that F(u — to)=0 is a constant solution, then du / dT = 0. The balancing solutions therefore correspond to the roots of the F(u) function. Yes, because our formula for the solution has been divided by F(u), it was presumed (7) we are not at equilibrium. In the example below, our solution formula (10) includes the balancing solution u(t) = real 0, similar to u0 = 0, but it is a lucky accident. Nonetheless, in the formula "general" solution, the balance solution does not appear (9). Usually, you have to be careful not to avoid balancing solutions while using this simple integration process.

**First Order Systems**

A first order system of ordinary differential equations has the general form

$$\frac{du_1}{dt} = F_1(t, u_1, \ldots, u_n), \qquad \cdots \qquad \frac{du_n}{dt} = F_n(t, u_1, \ldots, u_n). \quad (11)$$

The unknowns u1 (t), . . ., un (t) are scalar functions of the real variable t, which usually represents time. We shall write the system more compactly in vector form

$$\frac{d\mathbf{u}}{dt} = \mathbf{F}(t, \mathbf{u}), \quad (12)$$

where u(t) = ( u1 (t), . . . , un (t) )T , and F(t, u) = ( F1 (t, u1 , . . . , un ), . . . , Fn (t, u1 , . . . , un ) )T is a vector-valued function of n + 1 variables. A vector evaluation function u(t) means a solution for the differential equation, which is defined and continually distinguished by a < t < b interval and which also meets the differential equation at its definition interval. -- u(t) solution is used to set the curve C to R n, also known as a trajectory or system orbit.

In this analysis we will concentrate for these first order systems on initial value problems. The first general conditions are

$$u_1(t_0) = a_1, \qquad u_2(t_0) = a_2, \qquad \cdots \qquad u_n(t_0) = a_n, \quad (13)$$

or, in vectorial form,

$$\mathbf{u}(t_0) = \mathbf{a} \quad (14)$$

Here t0 is the initial specified time and the vector a = (a1, a2., .. a) T fixes the desired solution in its initial position. The initial conditions in favorable circumstances, as stated below, help to determine a solution to the differential equations — at least in nearby times. The following section discusses the general problems of life and the uniqueness of solutions.

A function is considered autonomous in differential equations if the right side is not directly dependent on time t.

$$\frac{d\mathbf{u}}{dt} = \mathbf{F}(\mathbf{u}). \quad (15)$$

The steady state fluid flows are an important class of first-order autonomous systems. The fluid velocity vector champ at the position u represents here F(u) = v. The solution u(t) describes the motion of a fluid particle which starts at a position t0 at the initial value problem. The differential equations tell us that the fluid speed fits the prescribed vector field at each point on the trajectory of the particle.

A balance solution is constant: u(t) oscillates u oscillates for all t. The derivative needs to vanish, du / dt = 0, and thus any solution of balance occurs as a way of solving the algebraic equation system

$$\mathbf{F}(\mathbf{u}^\star) = \mathbf{0} \quad (16)$$

prescribed by the vanishing of the right hand side of the system (15).

**Higher Order Systems**

Wide varieties of physical systems rely on the second and, occasionally, higher order derivatives of the unknown, nonlinear systems with difference equations. Yet an simple tool reduces an ordinary differential equation or structure to a first order equivalent function of any higher order. "Equivalent" implies that each first-order solution corresponds exclusively to a higher-order equation solution and vice versa. As a consequence, only first order structures need to be studied for all practical purposes. In fact, the vast majority of numerical solution algorithms are designed for first order systems and so an analogous first order method must be put to incorporate a higher order equation numerically.

In our discussion of the phase-plan approach to scalar second order equations we have already come across the main ideas

$$\frac{d^2u}{dt^2} = F\left(t, u, \frac{du}{dt}\right). \quad (17)$$

We introduce a new dependent variable $v = \frac{du}{dt}$. Since $\frac{dv}{dt} = \frac{d^2u}{dt^2}$ the functions u, v satisfy the equivalent first order system

$$\frac{du}{dt} = v, \qquad \frac{dv}{dt} = F(t,u,v). \tag{18}$$

On the contrary, if u(t) = (u(t), v(t))T is any solution to the first order, its first part u(t) specifies the scalar equation solution, which decides its equal value. For the first order method u (t0) = u0 v(t0) = v0; the simple initial conditions u(t0) = u0), u(t0) = v0; defining for second order the value of the solution and its first order derivative. A third order equation is also available

$$\frac{d^3u}{dt^3} = F\left(t, u, \frac{du}{dt}, \frac{d^2u}{dt^2}\right), \tag{19}$$

we set

$$v = \frac{du}{dt}, \qquad w = \frac{dv}{dt} = \frac{d^2u}{dt^2}.$$

The variables u, v, w satisfy the equivalent first order system

$$\frac{du}{dt} = v, \qquad \frac{dv}{dt} = w, \qquad \frac{dw}{dt} = F(t,u,v,w).$$

The general technique should now be clear.

**Example 2** The forced van der Pol equation

$$\frac{d^2u}{dt^2} + (u^2-1)\frac{du}{dt} + u = f(t) \tag{20}$$

arises in the modeling of an electrical circuit with a triode whose resistance changes with the current. It also arises in certain chemical reactions and wind-induced motions of structures. To convert the van der Pol equation into an equivalent first order system, we set v = du/dt, whence

$$\frac{du}{dt} = v, \qquad \frac{dv}{dt} = f(t) - (u^2-1)\,v - u, \tag{21}$$

is the equivalent phase plane system.

## 3. EXISTENCE, UNIQUENESS, AND CONTINUOUS DEPENDENCE.

There is, of course, no general analytical approach to address all differential equations. Nonetheless, scalar, non-autonomous ordinary differential equations, also fairly simple first order, cannot be solved similar. The solution of the Riccati equation, for instance

$$\frac{du}{dt} = u^2 + t \tag{22}$$

cannot be written in terms of elementary functions, although it can be solved in terms of Airy functions, [25].

$$\frac{du}{dt} = u^3 + t \qquad (23)$$

The Abel equation is even worse since its ultimate solution can not even be written in terms of standard special functions — while power series solutions can be tediously described on an individual basis. An important area of contemporary research [3] is the assumption that a given differential equation can be resolved as regards basic functions or recognized special functions. In this way, we can not avoid stating that symmetry is the most important class of exact methods for differential equations. The author's Graduate Level Monograph offers an introduction.

**Uniqueness and Smoothness**

The problem of singularity is as essential as nature. Are there more than one solution to the initial value problem? If so, we cannot estimate the potential conduct of the device from its present state with the differential equation. While continuity on the right side of the differential equation ensures that a solution exists, consistency of the solution to the original value problem is not enough sufficient. A basic illustration will help us understand the challenge.

Example 3 Consider the nonlinear initial value problem

$$\frac{du}{dt} = \frac{5}{3}\,u^{2/5}, \qquad u(0) = 0. \qquad (24)$$

Since the right hand side is a continuous function, Theorem assures us of the existence of a solution — at least for t close to 0. This autonomous scalar equation can be easily solved by the usual method:

$$\int \frac{3}{5}\,\frac{du}{u^{2/5}} = u^{3/5} = t + c, \qquad \text{and so} \qquad u = (t+c)^{5/3}.$$

Replacements with the original condition mean that the initial value problem is solved by c = 0, and thus u(t) = t 5/3.

On the other hand, the constant u(t) function 0 is a balance solution for the differential equation as the right hand side of the differential equation disappears at u = 0. In addition, the balance solution has the same initial value u(0) = 0. (There is an example of where the integration process does not recover the balance solution. We have developed therefore the initial value problem (24) with two separate solutions. Singularity is not

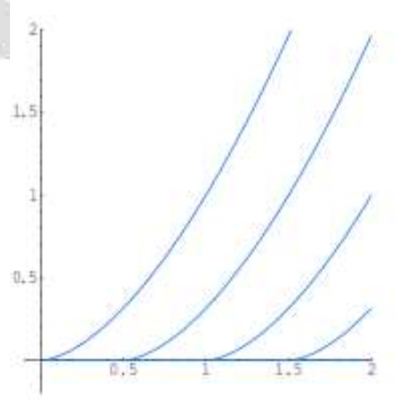


**Figure 2. Solutions to the Differential Equation** $\dot{u} = \frac{5}{3}\,u^{2/5}.$

valid! Worse yet, there are, in fact, an infinite number of solutions to the initial value problem. For any a > 0,

$$u(t) = \begin{cases} 0, & 0 \le t \le a, \\ (t-a)^{5/3}, & t \ge a, \end{cases} \qquad (25)$$

Everywhere, except at t = a, the function is differentiable. (Why?) In addition, both the differential equation and the initial condition are satisfied and then the problem with the initial value is defined. Some of the solutions in Figure 2 are presented.

Therefore we must enforce, beyond pure consistency, a tougher requirement in order to ensure that solutions are unique. In the references above we can find proof of the following theorem of simple uniqueness.

## 4. CONCLUSION

Non-linear structures can be directly solved such that the solution is typically based on a numerical scheme. The final part of the Chapter is devoted to the basic methods for initial value problems, from the simple Euler approach to the very famous fourth order Runge – Kutta method. Numerical systems do not always produce exact results, though, and we discuss briefly the form of rigid differential equations for numerical analysts which pose a major challenge. For any important theoretical understanding, it is not clear if numerical solutions (also those given by well-known packages) are to be used without a simple comprehension of the existence of solutions, equilibrium points and stability properties. This also helps to create a collection of non-linear concerns, of which one already knows one or more explicit theoretical solutions, by evaluating a computational scheme.

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