AN EXPLORATORY ON STUDY GRACE'S THEOREM AND ITS GENERALIZATIONS

Dr. Gyan Shekhar[1], Ab Waheed Lone[2]

[1]*Assistant Professor, Department of Mathematics, Bhagwant University, Ajmer, India*
[2]*Research Scholar, Department of Mathematics, Bhagwant University, Ajmer, India*

ABSTRACT

In this paper we are studding Study an Exploratory on Study Grace's Theorem and Its Generalizations.
As an application of Laguerrer's theorem, we will next introduce a result, known as the Grace theorem and which concerns further with the relative location of the zero of the two a polar polynomials. But before we state this result, we will first define the apolar polynomial.

Keyword: - *Polar, Polynomial, Degree, Area, Region etc.*

---

1. INTRODUCTION

We have obtained generalizations of (i), G (z) is treated as any polynomial of degree n and C with a circular region (consisting of 0) or convex complement and (ii) normalization. Not to be a circular area. Assuming g (z) as any polynomial of degree, n and C cannot be considered to be a spherical region (not containing 0) or a convex circular region. We have applied these generalizations to the study of the zeros (derived from two given polynomials) of some mixed polynomial, leading to some generalizations of Cézégo's theorem. Mathematis zeitschrift, incorporating circular regions (with a characteristic).

The Bôcher-Grace Theorem can be stated as follows: Let p be a third degree complex polynomial. Then there is a unique inscribed ellipse interpolating the midpoints of the triangle formed from the roots of p, and the foci of the ellipse are the critical points of p. Here, we prove the following generalization: Let p be an nth degree complex polynomial and let its critical points take the form

$\alpha+\beta\cos k\pi/n, k=1,\ldots,n-1, \beta\neq 0$.

Then there is an inscribed ellipse interpolating the midpoints of the convex polygon formed by the roots of p, and the foci of this ellipse are the two most extreme critical points of p: $\alpha\pm\beta\cos\pi/n$.

The fundamental theorem of algebra implies that each complex polynomial of degree n is numbered with multiplicity. Grace's theorem is a powerful tool that is used to obtain more precise information about the location of zeros of a polynomial. In particular it is useful when investigating how the null behaves under some transformations of the polynomial. Recall that the convex hull of a set S the C is the smallest convex set, in which S, that is, the square of all closed (circles), including the even-plane.

Suppose $f(z1, ..., zn)$ is a polynomial with complex coefficients, and that it is symmetric, i.e. invariant under permutations of the variables, and multi-affine, i.e. affine in each variable separately.

(Grace's Theorem). *Let $D \subset \mathbb{C}$ be a circular domain and let $f, g \in \mathbb{C}[z]$ be apolar polynomials of the same degree. If all zeros of f are contained in D, then g has at least one zero in D.*

Proof. Let $n = \deg(f) = \deg(g)$. Without loss of generality, we may assume that g is monic and write $g = \prod_{k=1}^{n}(z - \mu_k)$. Then

$$[f, g]_n = \mathrm{Sym}_n(f)(\mu_1, \ldots, \mu_n)$$

by By Walsh's theorem, at least one of $\mu_1, \ldots, \mu_n$ must lie in D, as claimed.

2. EXAMPLES

Example Let $q = z^2 + \beta z + \gamma$. Two points $z, w \in \mathbb{C}$ have the same image under q if and only if $z + w = -\beta$. This implies

$$q^{-1}(q_\circ(D)) = \{z \in D \mid -(z + \beta) \in D\} = D \cap -(D + \beta).$$

and $q_\circ(D)$ is the image of that region under q. In particular, if $D = \mathbb{D}$ is the open unit disk, then $q_\circ(\mathbb{D})$ is non-empty if and only if $|\beta| < 2$.

For example, take $q = z^2 + \frac{1}{2}z$. The image of the unit circle under q is the real quartic curve $\{z = x + iy \mid u(x, y) = 0\}$ in the complex plane, where

$$u(x, y) = 4x^4 + 4y^4 + 8x^2y^2 - 9x^2 - 9y^2 - 2x + 3.$$

The preimage of this curve under q consists of the unit circle and the shifted unit circle $\{z \in \mathbb{C} \mid |z + \frac{1}{2}| = 1\}$. The region $q_\circ(\mathbb{D})$ and its preimage are shown in Fig. 1.

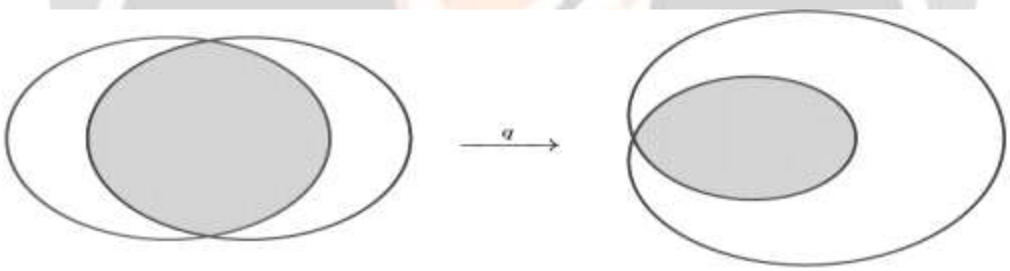

FIGURE 1. The region $q_\circ(\mathbb{D})$ and its preimage for $q = z^2 + \frac{1}{2}z$

This can be verified by noting that p(x) can be factored as (x2 − 1)(x2 + x + 1), where the first factor has the roots −1 and 1, and second factor has no real roots. This last assertion results from the quadratic formula, and also from theorem, which gives the sign sequences (+, –, –) at −∞ and (+, +, –) at +∞.

3. THEOREM AND LEMMA

THEOREM If

$$P(z) = \sum_{j=0}^{n} C(n, j) A_j z^j, \qquad A_0 A_n \neq 0$$

and

$$Q(z) = \sum_{j=0}^{m} C(m, j) B_j z^j, \qquad B_0 B_m \neq 0$$

are two polynomials degree n and m respectively $m \leq n$ such that

$$c(m, 0) B_0 A_n - c(m, 1) B_1 A_{n-1} + \cdots + (-1)^m c(m, m) B_m A_{n-m} = 0,$$

then the following holds.

(i) $Q(z)$ has all its zeros in $|z-c| \geq r$, then $P(z)$ has at least one zero in $|z-c| \geq r$.
(ii) $P(z)$ has all its zeros in $|z-c| \leq r$, then $Q(z)$ has at least one zero in $|z-c| \geq r$.

For the proof of the Theorem 4.4, we need the following lemma, which is a generalization of a result due to Markovitch [37, p. 64].

LEMMA Let

$$P(z) = \sum_{j=0}^{n} C(n,j) A_j z^j$$

and

$$Q(z) = \sum_{j=0}^{m} C(n,j) B_j z^j$$

be two polynomials of degree n and m, respectively $n \leq m$. If we form

$$U(z) = \sum_{j=0}^{n} (-1)^j P^{(n-j)}(z) Q^{(j)}(z),$$

then

$$U(z) = n! \sum_{j=0}^{n} (-1)^j c(m,j)\, A_{n-j} B_j\,.$$

PROOF OF LEMMA Since $P(z)$ and $Q(z)$ are two polynomials of degree n and m respectively, we have

$$P^{(n+1)}(z) = 0 \quad \text{and} \quad Q^{(m+1)} = 0.$$

Now we can write

$$P(z) = \sum_{j=0}^{n} C(n,j) A_j z^j = \sum_{j=0}^{n} \frac{P^{(j)}(0)}{j!} z^j$$

and

$$Q(z) = \sum_{j=0}^{n} C(m,j) B_j z^j = \sum_{j=0}^{m} \frac{Q^{(j)}(0)}{j!} z^j,$$

from which it follows that,

$$P^{(j)}(0) = C(n,j) j!\, A_j, \qquad j = 0,1,\cdots,n, \tag{3.25}$$

and

$$Q^{(j)}(0) = C(m,j) j!\, B_j, \qquad j = 0,1,\cdots,m. \tag{3.26}$$

Now

$$U'(z) = \sum_{j=0}^{n} (-1)^j P^{(n-j+1)}(z) Q^{(j)}(z)$$

$$+ \sum_{j=0}^{n} (-1)^j P^{(n-j)}(z) Q^{(j+1)}(z),$$

so that by (3.24), we have

$$U'(z) = \sum_{j=1}^{m} (-1)^j P^{(n-j+1)}(z) Q^{(j)}(z) + \sum_{j=0}^{m-1} (-1)^j P^{(n-j)}(z) Q^{(j+1)}(z)\,,$$

$$= \sum_{j=0}^{m-1} (-1)^{j+1} P^{(n-j)}(z) Q^{(j+1)}(z) + \sum_{j=0}^{m-1} (-1)^{j} P^{(n-j)}(z) Q^{(j+1)}(z),$$

$$= 0,$$

therefore $U(z)$ is constant and thus

$$U(z) = U(0) = \sum_{j=0}^{m} (-1)^j P^{(n-j)}(0) Q^{(j)}(0).$$

Using (3.25) and (3.26), we obtain

$$U(z) = n! \sum_{j=0}^{n} (-1)^j c(m,j)\, A_{n-j} B_j.$$

This proves the lemma.

4 GENERALIZATIONS

Grace sequences have been generalized in two directions. To define each polynomial in the sequence, Grace used the negative of the remainder of the Euclidean division of the two preceding ones. The theorem remains true if one replaces the negative of the remainder by its product or quotient by a positive constant or the square of a polynomial. It is also useful (see below) to consider sequences where the second polynomial is not the derivative of the first one.

A *generalized Grace* is a finite sequence of polynomials with real coefficients

$P_0, P_1 \ldots\ldots\ldots P_m$

such that

- the degrees are decreasing after the first one: $\deg P_1 < \deg P_{i-1}$ for $i = 2, ..., m$;
- P_m does not have any real root or does not changes of sign near its real roots.
- if $P_i(\xi) = 0$ for $0 < i < m$ and ξ a real number, then $P_{i-1}(\xi)\, P_{i+1}(\xi) < 0$.

The last condition implies that two consecutive polynomials do not have any common real root. In particular the original Grace sequence is a generalized Grace sequence, if (and only if) the polynomial has no multiple real root (otherwise the first two polynomials of its Grace sequence have a common root).

When computing the original Grace sequence by Euclidean division, it may happen that one encounters a polynomial that has a factor that is never negative, such a x^2 or x^2+1. In this case, if one continues the computation with the polynomial replaced by its quotient by the nonnegative factor, one gets a generalized Grace sequence, which may also be used for computing the number of real roots, since the proof of Grace's theorem still applies (because of the third condition). This may sometimes simplify the computation, although it is generally difficult to find such nonnegative factors, except for even powers of *x*.

5. APPLICATION

Generalized Grace allow counting the roots of a polynomial where another polynomial is positive (or negative), without computing these root explicitly. If one knows an isolating interval for a root of the first polynomial, this allows also finding the sign of the second polynomial at this particular root of the first polynomial, without computing a better approximation of the root.

Let P(x) and Q(x) be two polynomials with real coefficients such that P and Q have no common root and P has no multiple roots. In other words, P and P'Q are coprime polynomials. This restriction does not really affect the generality of what follows as GCD computations allows reducing the general case to this case, and the cost of the computation of a Graces is the same as that of a GCD.

Let W(a) denote the number of sign variations at a of a generalized Gracesequence starting from P and P'Q. If a < b are two real numbers, then W(a) – W(b) is the number of roots of P in the (a,b) such that Q(a) > 0 minus the number of roots in the same interval such that Q(a) < 0. Combined with the total number of roots of P in the same interval given by Sturm's theorem, this gives the number of roots of P such that Q(a) > 0 and the number of roots of P such that Q(a) < 0.[1]

REFERENCES

[1]. A. Aziz, A New proof of Laguerre's theorem about the zeros of polynomials, Bull. Austral. Math. Soc. 33(1986), 131-138.

[2]. A. Aziz, On the location of the zeros of certain composite polynomials, Pacific J. Math. 118(1985), 17-26.

[3]. A. Aziz, On the zeros of composite polynomials, Pacific J. Math. 103(1982), 1-7.

[4]. A. Aziz and Q. Aliya, Estimates for the modulii of the zeros of a polynomial, Math. Ineq. Appl. 9, (2006), 107-116.

[5]. A. Aziz and Q. G. Mohammad, Zero-Free regions for polynomials and some generalizations of Eneström-Kakeya Theorem, Canad. Math. Bull. Vol. 27, No. 3(1984), PP. 263-272.

[6]. A. Aziz and Q. G. Mohammad, On Zeros of Certain Class of Polynomials and Related Analytic Functions, J. Math. Anal. Appl. 75 (1980), 495-502.

[7]. A. Aziz and N. A. Rather, A compact generalization of Walsh's two circle theorem for polynomials and rational functions, Indian j. Pure. Appl. Math. 27(8): 785-790, August 1996.

[8]. A. Aziz and W.M. Shah, On the location of zeros of polynomials and related analytic functions, non-linear studies 6(1999), 91-101.

[9]. N. Anderson, E. B. Saff, R. S. Varga, An extension of Eneström- Kakeya theorem and its sharpness, SIAM Math. Anal. 12(1981), 10-22.

[10]. N. Anderson, E. B. Saff, R. S. Varga, On the Eneström-Kakeya theorem and its sharpness, Linear Algebra and its applications. 28(1979), 5-16.

[11]. A. Aziz and B. A. Zargar, On the zeros of polynomials, Proc. Indian Acad. Sci. (Math. Sci), 106(1996), 127-132.

[12]. A. Aziz and B. A. Zargar, Some extensions of Eneström- Kakeya theorem, Glasnik Mathematici, 31(1996), 239-244.

[13]. A. Aziz and B. A. Zargar, Some refinements of Eneström-Kakeya theorem, Analysis in theory and Applications 23(2007), 129-137.