# APPLICATION OF HYBRID METHODS TO SOLVING VOLTERRA INTEGRAL EQUATION WITH THE FIXED BOUNDARES

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**ABSTRACT**

*In the middle of XX century the scholars began construct the methods which have the best characteristics of the one and multi-step methods with the constant coefficients. And in the 1955 years there appears the works of Gear and Butcher dedicated investigation of the hybrid method. Here we want to show that how this theory developed and compared them with the known methods. Constructed the hybrid method with the higher order of accuracy and illustrated them by the model equation. For application of hybrid methods, here proposed the simple algorithms with the order of accuracy* $p \le 10$ *in the case, when the amount of the used mesh points is equal to 2 or the order of the difference methods variable satisfies the condition* $k = 1$.

**Keyword:** - *hybrid methods, Volterra integral equation, accuracy and stability of numerical methods, ODE, multistep methods with the second derivative*

## 1. TITLE-1

Many problems of natural sciences reduced to solving integral equations with the variable boundaries which in linear case was fundamentally investigated by Vito Volterra. Many scholars have investigated the numerical solution of the Volterra integral equations. But they in basically use the quadrate methods or its modification. But here proposed a new way for construction the numerical methods for solving the integral equation with the variable boundaries.

Let us to consider the following integral equation

$$y(x) = f(x) + \int_{x_0}^{x} K(x, s, y(s))ds . \qquad (1)$$

Sometimes integral equation (1) called the integral equation of Volterra-Urison type.

This convention is due to the fact that equation (1) in the linear case was thoroughly investigated by Volterra, and at a sufficiently high level was studied such equations occurrence in practical problems (see, e.g. [1], [2]). Note that the singular integral equation with a variable boundary in a particular form was studied for the first time by Abel (see, e.g., [1, p.12]). Given that even in the linear case, the exact solution to equation (1) is not always possible, therefore many experts have used approximation methods to solve it (see, e.g., [3] - [7]).

Here we suppose that the Volterra integral equation (1) has the unique solution determined on the interval $[x_0, X]$, for investigation of the numerical solution of equation of (1), assume that the kernel $K(x, z, y)$ of the integral equation define in the domain $\overline{G} = \{x_0 \le s \le x + \varepsilon \le X, |y| \le b\}$, where it has partial derivatives up to some order $p$, inclusively. But the given sufficiently smooth function $f(x)$ has determined on the interval $[x_0, X]$. For determining the approximate values of solution of equation (1), we divide the interval

$[x_0,\ X]$ into $N$ equal parts by the mesh points $x_i = x_0 + ih\ (i = 0,1,2,...,N)$. Denote by $y_i$ the approximate and $y(x_i)$ exact values of solution of equation (1) at the mesh points $x_i\ (i = 0,1,2,...,N)$.

As is known one of the popular methods for solving equation (1) is the quadrature method. The quadrature method which applies to solution of equation (1) can write as follows (see [2]):

$$y(x_n) = f(x_n) + h\sum_{j=0}^{n} \bar{a}_j K(x_n, x_j, y(x_j)) + R_n, \tag{2}$$

where $R_n$ is the remainder term of the quadrature method and $\bar{a}_j\ (j = 0,1,...,n)$ are real numbers; these numbers called the coefficients of the quadrature method. By discarding the remainder term, we obtain the following method:

$$y_n = f_n + h\sum_{j=0}^{n} \bar{a}_j K(x_n, x_j, y_j) \quad (n = 1,2,3,...), \quad y_0 = f(x_0), \tag{3}$$

which called the quadrature method with the variable boundary.

Remark that the method (3) for $\bar{a}_n \neq 0$ is implicit, but this method for $\bar{a}_n = 0$ is explicit. And in the result of using method of (3), by increasing of the values of the quantity $n$, the amount of computing functions $K(x,s,y)$ is increases, also. Consequently, at each step the amount of calculation work is increases, which is a major disadvantage of quadrature methods. To eliminate this drawback of method (3), some authors have proposed the use of methods such as the following $k$-step method with constant coefficients (see for example [8] - [11]):

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i f(x_{n+i}, y_{n+i}), \tag{4}$$

It is known that if the method (4) is stable then the following is holds:

$$p \leq 2[k/2] + 2. \tag{5}$$

Here the degree of the method (4) quantity $p$ is order of the accuracy and $k$ is the order of the difference method (4).

For the construction the stable methods with the degree $p > k + 2$, the scholars are use modification of the method (4) in with resulted appears the forward-jumping methods etc. Some of these scholars proposed to change the method (4) by the following:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i y'_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i y''_{n+i} \tag{6}$$

Here the coefficients $\alpha_i, \beta_i, \gamma_i\ (i = 0,1,...,k)$ are the some real numbers and $\alpha_k \neq 0$.

It is clear that for generalization of the Runge-Kutta and Adams methods one may use the different ways, one of which can be written as the follows:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \gamma_i y'(x_{n+i} + \nu_i h), \quad (|\nu_i| < 1; i = 0,1,2,...,k) \tag{7}$$

Here quantities are the hybrid points.

Remark that for function of the methods Runge-Kutta and Adams. We rewrite the method (6) as the following:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i y'_{n+i} + h\sum_{i=0}^{k} \gamma_i y'_{n+i+\nu_i} \tag{8}$$

If we continue the above proposed way, then the method (5) can rewrite as:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h(\sum_{i=0}^{k} \beta_i y'_{n+i} + \sum_{i=0}^{k} \gamma_i y'_{n+i+\nu_i}) + h^2(\sum_{i=0}^{k} \beta_i y''_{n+i} + \sum_{i=0}^{k} \gamma_i y''_{n+i+l_i}) \tag{9}$$

It is not difficult to understand that the method (9) is more accurate, than the known methods.

Method (4) generalized so that every following method is more accurate than the previous one. For example, in family methods of type (8) there are stable methods that are more accurately than the methods of the type (6) and (7). Therefore research of methods (8) and (9) is more promising than the above methods of research. For the objectivity, we note that hybrid methods are more accurate than the known. However, using of the hybrid method is a difficult process. Therefore, construction of algorithms for application of hybrid methods is sometimes more difficult process than construction of the method itself. In the scientific literature, some authors such method called fractional steps.

## 2. APPLICATION OF THE METHOD OF UNDETERMINED COEFFICIENTS TO THE STUDY OF HYBRID METHODS

For construction of hybrid methods we can use different schemes, in resulting of this, get methods with the different properties. It is known that the properties of numerical methods depend on the values of the coefficients in the formulas (5) - (9). Previously, the above method applies to solution of the following initial value problem:

$$y' = F(x, y), y(0) = 0. \tag{10}$$

In these studies, we used the method of undetermined coefficients for determination of coefficients in the formulas (5) - (9). Therefore, we will try in construction of hybrid methods for solving the equation (1) using the method of undetermined coefficients. Note that one of the most popular methods for finding solutions of Volterra integral equations is quadrature method. But as shown above, when using the quadrature methods the volume of computations increases with the number of quantity of the points used in constructing them. In this regard, here we offer apply to solution of equation (1), the following multi-step method

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{i=0}^{k} \alpha_i F_{n+i} + h \sum_{j=0}^{k} \sum_{i=0}^{k} \beta_i^{(j)} K(x_{n+j}, x_{n+i}, y_{n+i}). \tag{11}$$

Note that from the formula (11) we can obtain implicit and explicit methods, as well as forward-jumping methods. Here we want by using the above schemes construct hybrid methods for solving of the equation (1). The method (11) has a straight connection with the method (5). Indeed, if in the equation (1), the kernel of the integral, the function $K(x,s,y)$ the next: $K(x,s,y) = F(s,y)$ , then solution of the equation (1) coincides with solution of the problem (10). Using a similar connection between integral and differential equations, consider construction of hybrid methods.

Here, to solve equation (1), we use the finite-difference method, which can be written as follows:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} (\beta_i y'_{n+i} + \hat{\beta}_i y'_{n+i+m_i}) \quad (|m_i| < 1;\ i = 0, 1, ..., k). \tag{12}$$

On the first let us consider any bounders for the coefficients of the method (2).

A: The coefficients $\alpha_i, \beta_i, \gamma_i, \nu_i$ $(i = 0,1,2,\ldots,k)$ are some real numbers, moreover, $\alpha_k \neq 0$ .

B: Characteristic polynomials

$$\rho(\lambda) \equiv \sum_{i=0}^{k} \alpha_i \lambda^i, \ \ \delta(\lambda) \equiv \sum_{i=0}^{k} \beta_i \lambda^i; \ \gamma(\lambda) \equiv \sum_{i=0}^{k} \gamma_i \lambda^{i+\nu_i} \ \ (\beta_i = \gamma_i).$$

have no common multipliers different from the constant.

C: $\sigma(1) + \gamma(1) \neq 0$ and $p \geq 1$.

In particular, for $\hat{\beta}_i = 0 \ \ (i = 0,1,2,\ldots,k)$, from the method of (12) we obtain an ordinary $k$ -step method with the constant coefficients.

## 3. ON A WAY TO CONSTRUCTION HYBRID METHOD OF TYPE

Consider the construction of the methods of type (12) for $\hat{\beta}_i = 0 \ (i = 0,1,2,\ldots,k)$. To this end, we construct a formula for calculating the values of the function $y'(x)$. Suppose that by some methods the solution of the equations is found after taking into account that in equation (1) one identity is obtained. Then, from this identity we can write:

$$y'(x) = f'(x) + \int_{x_0}^{x} K'_x(x, s, y(s))ds \ . \tag{13}$$

Finding the values of the quantities $y'_{n+k}$ by means of formulas (13) is reduced to the calculation of the following integral:

$$\int_{x_0}^{x_m} K(x_m, s, y(s))ds. \tag{14}$$

If quadrature formula (3) is applied to an evaluation of integral (14), then we obtain the integral sum of the variable boundaries. But for the application of the method (12) to solving equation (1), we give the known values $y_{n+k-1},\ y'_{n+k-1}$ and we want to find the relationship between these values with the values of $y_{n+k},\ \ y'_{n+k}$ which are unknown.

The computation in the segment $[x_0, x_{n+1}]$ the integral with the variable boundary has been reduced to calculations that in the segment $[x_n, x_{n+1}]$. Moreover, this reduction was accomplished without increasing the computational work by increasing the values of the variable $n$. Note that this type of scheme is applied to the determination of the relationship between the variables $y'_{n+1}$ and $y'_n$.

It is known that for a sufficiently smooth function, the following is holds:

$$\sum_{i=0}^{k} \hat{\alpha}_i z_{n+i} = hz'(\xi_{n+1}), \tag{15}$$

here $\hat{\alpha}_i \ \ (i = 0,1,..., k)$ are real numbers.

After carrying out same operations, we obtain:

$$\sum_{i=0}^{k} \alpha_i y'_{n+i} = \sum_{i=0}^{k} \alpha_i f_{n+i} + \sum_{i=0}^{k} \alpha_i a_{n+i} + h\sum_{j=0}^{k}\sum_{i=0}^{k} \gamma_i^{(j)} K'_x(x_{n+j}, x_{n+i}, y_{n+i}). \tag{16}$$

here $a(x) = K(x, x, y(x))$

Depending on the needs of the user, the coefficients in formulas (14) can be given different meanings. For this purpose, equations (13) can be rewritten as follows:

$$y'(x) = f'(x) + a(x) + \frac{1}{h}\sum_{i=0}^{k} \hat{\alpha}_i \int_{x_0}^{x} K(x - ih, s, y(s))ds + R_1, \tag{17}$$

Here, $\hat{\alpha}_i \ (i = 0,1,..., k)$ are real numbers, but $R_1$ is the remainder term. By replacing the integrals in formulas (17) with an integral sum and discarding the remaining terms, we obtain methods to calculate the values of the quantity $y'_{n+k}$. However, in this case, the coefficients impose the following additional conditions. It is not difficult to proof that for the finding of the coefficients of the method (12) may be use the following system of the nonlinear algebraic equation:

$$\begin{gathered} \sum_{i=0}^{k} \alpha_i = 0,\ \ \sum_{i=0}^{k} i\alpha_i = \sum_{i=0}^{k} \beta_i + \sum_{i=0}^{k} \gamma_i, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \\ \sum_{i=0}^{k} \frac{i^p}{p!}\alpha_i = \sum_{i=0}^{k} \frac{i^{p-1}}{(p-1)!}\beta_i + \sum_{i=0}^{k} \frac{l_i^{p-1}}{(p-1)!}\gamma_i . \end{gathered} \tag{18}$$

By using the solution of algebraic equations can construct methods having different properties. From the viewpoint of the application of numerical methods, the most significant property is the stability and accuracy. The accuracy of these methods depends on the number of equations in the system (18). However, when the number of equations in the system (18), there is some difficulty in finding the solution of the system (18). The above mentioned difficulty is not finding the solution of linear algebraic equations. Note that we constructed methods maximum accuracy only in some particular cases, the following are some of the mood of them:

$$y_{n+1} = y_n + h(y'_{n+1} + y'_n)/12 + 5h(y'_{n+1/2-\alpha} + y'_{n+1/2+\alpha})/12, (\alpha = \sqrt{5}/10) \quad (19)$$

$$y_{n+1} = (11y_n + 8y_{n+1/2})/19 + h(10y'_n + 57y'_{n+1/2} + 24y'_{n+1} - y_{n+3/2})/114 \quad (20)$$

$$y_{n+1} = y_n + h(64y'_{n+1} + 98y'_{n+1/2} + 18y'_n)/360 + h(18y'_{n+1/2+\beta/2} + 98y'_{n+1/2} + 64y'_{n+1/2-\beta/2})/360. \quad (21)$$

$$y_{n+2} = y_n + h(5y'_{n+1+\alpha} + 8y'_{n+1} + 5y'_{n+1-\alpha})/9, (\alpha = \sqrt{15}/5), \quad (22)$$

As can be seen from the above stable hybrid methods are more accurate than corresponding methods of Runge-Kutta and Adams.

In a specific example, we show that if we use the Simpson method to solve it, the result obtained by step $h/2$ is better than the result in the step $h$.

To this end, consider the following tasks:

$$y' = \cos x, y(0) = 1, x \in [0,1],$$

exact solution is written as: $y = \sin x$.

The result at the step $h = 0,1$ to place in the following:

Table 1.

| Value of the variable $x$ | Error for the Simpson method by the step $h/2$ | Error for the Simpson method by the step $h$ |
|---|---|---|
| 0.10 | 0.34E-08 | 0.11E-06 |
| 0.20 | 0.69E-08 | 0.10E-06 |
| 0.30 | 0.10E-07 | 0.21E-06 |
| 0.40 | 0.13E-07 | 0.21E-06 |
| 0.50 | 0.16E-07 | 0.31E-06 |
| 0.60 | 0.19E-07 | 0.30E-06 |
| 0.70 | 0.22E-07 | 0.39E-06 |
| 0.80 | 0.24E-07 | 0.38E-06 |
| 0.90 | 0.27E-07 | 0.46E-06 |
| 1.00 | 0.29E-07 | 0.44E-06 |

As seen from Table 1, the results obtained by the following method of Simpson

$$y_{n+1} = y_n + h(y'_{n+1} + 4y'_{n+1/2} + y'_n)/6$$

are accurate.

Now consider the application of the next hybrid method of type (7):

$$y_{n+1} = y_n + h(f_{n+l_0} + f_{n+1+l_1})/2, \ (y'(x) = f(x, y(x))), (l_1 = 1/2 + \sqrt{3}/6; \ l_0 = 1/2 - \sqrt{3}/6). \quad (24)$$

To solving following examples:

1. $y' = \cos x, \ y(0) = 0, \ x \in [0,1]$. Exact solution: $y(x) = \sin x$.
2. $y' = \lambda y, \ y(0) = 1, \ x \in [0,1], \ \lambda = \pm 1$. Exact solution: $y(x) = \exp(\lambda x)$.

For the calculating of the values $y_{n+1}$ and $y_{n+1-l}$ may be used the methods:

$$y_{n+l} = y_n + lhf_n,$$

$$\hat{y}_{n+l} = y_n + lh(f_n + \bar{f}_{n+l})/2,$$

$$(\bar{f}_m = f(x_m, \bar{y}_m), \ m = 0,1,2,\dots).$$

Repeat these schemes for $l := 1 - l$.

The results of calculations accommodated in Table 2 and Table 3.

Table 2

| Step size | Variable $x$ | Error for the example 1 | Error for the example 2 ($\lambda=1$) |
|---|---|---|---|
| $h=0.10$ | 0.20 | 0.11E-06 | 0.61E-05 |
| | 0.30 | 0.10E-06 | 0.78E-05 |
| | 0.40 | 0.21E-06 | 0.15E-04 |
| | 0.50 | 0.21E-06 | 0.19E-04 |
| | 0.60 | 0.31E-06 | 0.28E-04 |
| | 0.70 | 0.30E-06 | 0.34E-04 |
| | 0.80 | 0.39E-06 | 0.46E-04 |
| | 0.90 | 0.38E-06 | 0.56E-04 |
| | 1.00 | 0.46E-06 | 0.71E-04 |

Table 3

| Step size | Variable $x$ | Error for the example 1 | Error for the example 2 ($\lambda=1$) |
|---|---|---|---|
| $h=0.05$ | 0.10 | 0.34E-08 | 0.36E-06 |
| | 0.20 | 0.69E-08 | 0.81E-06 |
| | 0.30 | 0.10E-07 | 0.13E-05 |
| | 0.40 | 0.13E-07 | 0.19E-05 |
| | 0.50 | 0.16E-07 | 0.27E-05 |
| | 0.60 | 0.19E-07 | 0.36E-05 |
| | 0.70 | 0.22E-07 | 0.47E-05 |
| | 0.80 | 0.24E-07 | 0.59E-05 |
| | 0.90 | 0.27E-07 | 0.74E-05 |
| | 1.00 | 0.29E-07 | 0.91E-05 |

## 4. CONSTRUCTION AND APPLICATION THE METHODS WITH THE HIGHER ORDER OF ACCURACY TO SOLVING NONLINEAR VOLTERRA INTEGRAL EQUATION

As noted above, one of the main issues in modern computational mathematics is the construction of more accurate methods with extended stability of the region, and as an example of such methods is proposed a method (6). Here, continued this idea, are construct a hybrid method with the second derivative, which can be represented in the form (9), where the coefficients $\alpha_i, \beta_i, \hat{\beta}_i, \gamma_i, \hat{\gamma}_i$ $(i=\overline{0,k})$ are some real numbers, and besides $\alpha_k \neq 0$ is hold. We assume that the coefficients of the method (9) satisfy the following conditions:

A. The coefficients $\alpha_i, \beta_i, \hat{\beta}_i, \gamma_i, \hat{\gamma}_i$ $(i=0,1,2,\ldots,k)$ are real numbers and $\alpha_k \neq 0$.

B. The sets of the roots of the characteristic polynomials

$$\rho(\lambda) \equiv \sum_{i=0}^{k} \alpha_i \lambda^i, \;\; \vartheta(\lambda) \equiv \sum_{i=0}^{k} \beta_i \lambda^i, \;\; \hat{\vartheta}(\lambda) \equiv \sum_{i=0}^{k} \hat{\beta}_i \lambda^{i+\nu_i},$$

$$\gamma(\lambda) \equiv \sum_{i=0}^{k} \gamma_i \lambda^i, \;\; \hat{\gamma}(\lambda) \equiv \sum_{i=0}^{k} \hat{\gamma}_i \lambda^{i+l_i}.$$

Have not common factor different from the constant.

C. The following holds:

$$\rho'(1) = \vartheta(1) + \hat{\vartheta}(1) \neq 0, \; \rho''(1) \neq 0 \text{ and } p \geq 2.$$

Note that the accuracy of the method (9) is determined by similar to the concept of the degree of the method (9), which can be formulated in the following form:

Definition 1. The quantity $p$ is called the degree of the method (9), if holds the following:

$$\sum_{i=0}^{k}[\alpha_i y(x+ih)-h[\beta_i y'(x+ih)+\hat{\beta}_i y'(x+ih+\nu_i h)]-h^2[\gamma_i y''(x+ih)+\hat{\gamma}_i y''(x+ih+l_i h)]=O(h^{p+1}),\ h\to 0. \quad (25)$$

However, if $\rho'(1)=0$, then the degree of method (9) is determined by using similar asymptotic relations to (25), where $p\geq 1$ and $\rho''(1)\neq 0$.

Here, also we use the method of undetermined coefficients for the determination of the coefficients of the method (9). We must take into account that the basic properties of numerical methods are determined by the values of their coefficients, and thus, we consider the determination of the values of the coefficients of method (9). For this purpose, in method (9) the approximate values of the function $y(x)$ will be replaced by the exact values. Then, we have:

$$\sum_{i=0}^{k}\alpha_i y(x+ih)=h\sum_{i=0}^{k}(\beta_i y'(x+ih)+\hat{\beta}_i y'(x+(i+\nu_i)h))+h^2\sum_{i=0}^{k}(\gamma_i y''(x+ih)+\hat{\gamma}_i y''(x+(i+l_i)h))+R_n, \quad (26)$$

where $R_n$ is the remainder term of method (9). If we suppose that method (9) has the degree $p$, then from equation (26) we have $R_n=O(h^{p+1})$.

To find the values of the quantities $\alpha_i,\beta_i,\hat{\beta}_i,\gamma_i,\hat{\gamma}_i,\nu_i,l_i \quad (i=0,1,2,\ldots,k)$, we use the method of undetermined coefficients, i.e., in equation (26) we use Taylor expansions of the functions $y(x),y'(x)$ and $y''(x)$. Consider the following expansions of these functions:

$$y(x+ih)=y(x)+ihy'(x)+\frac{(ih)^2}{2!}y''(x)+...+\frac{(ih)^p}{p!}y^{(p)}(x)+O(h^{p+1}),$$

$$y^j(x+ih)=y^{(j)}(x)+ihy^{(j+1)}(x)+\frac{(ih)^2}{2!}y^{(j+2)}(x)+\ldots+\frac{(ih)^{p-j}}{(p-j)!}y^{(p)}(x)+O(h^{p+1-j}),$$

$$\begin{aligned}y^j(x+(i+m_i)h)&=y^{(j)}(x)+(i+m_i)hy^{(j+1)}(x)+\frac{((i+m_i)h)^2}{2!}y^{(j+2)}(x)+\ldots+\\&+\frac{((i+m_i)h)^{p-j}}{(p-j)!}y^{(p)}(x)+O(h^{p+1-j}),\end{aligned} \quad (27)$$

where the quantities $j$ and $m_i\ (i=0,1,2,...,k)$ receive the following values: $j=1,2;\ m_i=\nu_i$ or $m_i=l_i\ (i=0,1,2,...,k)$.

Assume that method (9) has the degree $p$ and consider determining the values of its coefficients. By using expansions (27) in the (25), we must assume that the method (9) has the degree $p$. In this case its coefficients satisfy the following conditions:

$$\sum_{i=0}^{k}\alpha_i=0;\ \sum_{i=0}^{k}(\beta_i+\hat{\beta}_i)=\sum_{i=0}^{k}i\alpha_i,$$

$$\sum_{i=0}^{k}(\gamma_i+\hat{\gamma}_i)+\sum_{i=0}^{k}(i\beta_i+(i+\nu_i)\hat{\beta}_i)=\frac{1}{2}\sum_{i=0}^{k}i^2\alpha_i, \quad (28)$$

$$(m-1)\sum_{i=0}^{k}(i^{m-2}\gamma_i+(i+l_i)^{m-2}\hat{\gamma}_i)+\sum_{i=0}^{k}(i^{m-1}\beta_i+(i+\nu_i)\nu_i^{m-1}\hat{\beta}_i)=\frac{1}{m}\sum_{i=0}^{k}i^m\alpha_i$$

$$(m=3,4,\ldots,p).$$

The resulting correlation is a homogeneous system of nonlinear algebraic equations. Obviously, system (28) always has a zero (trivial) solution that is not useful for the construction of the methods addressed in this paper. Therefore, to investigate the nonzero (nontrivial) solutions of nonlinear system (28), in which we observe that the number of equations is equal to $p+1$, but the number of unknowns is equal to $7k+7$. It can be assumed that systems (28) will have a nontrivial solution for the values

$$p \le 7k+5.$$

Thus, we see that by using solution of the system (28), we can construct methods of type (9) with the degree

$$p \le 7k+5. \tag{29}$$

It is not consequently that there exist the methods of type (9) with the degree $p > 7k+5$. However, methods, constructed by us obey the rule (25).

Consider the following one-step method:

$$y_{n+1} = y_n + h(64y'_{n+1} + 98y'_{n+\frac{1}{2}} + 18y'_n)/360 + h(18y'_{n+l_2/2} + 98y'_{n+1/2} + 64y'_{n+l_0/2})/360 \tag{30}$$

here $l_2 = 1+\sqrt{21}/14$, $l_1 = 1$, $l_0 = 1-\sqrt{21}/14$.

This method is stable and has a degree $p=8$. Note that this method is not obtained from the formula (9), because at $k=1$ in the construction the method of the type (9) is used only 2 mesh points. But in the method (30) participate three mesh points. Therefore, the method (30), in particular is not contained in the class of the method (9).For simplicity, we consider a special case and set $k=1$. Then, by using of the method of type (9) we obtain:

$$\begin{aligned} y_{n+1} = y_n + h(\beta_0 y'_n + \beta_1 y'_{n+1}) + h(\hat{\beta}_0 y'_{n+\nu_0} + \hat{\beta}_1 y'_{n+\nu_1}) + h^2(\gamma_0 y''_n + \gamma_1 y''_{n+1}) + \\ + h^2(\hat{\gamma}_0 y''_{n+l_0} + \hat{\gamma}_1 y''_{n+l_1}). \end{aligned} \tag{31}$$

In applications of method (31) to solve certain problems, it is necessary to determine the values of the quantities $y_{n+\nu_i}$ and $y_{n+l_i}$ $(i=0,1)$. Some authors prefer to construct methods of type (9) for which the following condition holds: $\nu_i = l_i$ $(i=0,1)$. By using the conditions $\nu_i = l_i$ $(i=0,1)$ one can be obtained from system (28), we obtain the following:

$$\begin{gathered} \beta_1 + \beta_0 + \hat{\beta}_1 + \hat{\beta}_0 = 1, \\ \gamma_1 + \gamma_0 + \hat{\gamma}_1 + \hat{\gamma}_0 + \beta_1 + l_1\hat{\beta}_1 + l_0\hat{\beta}_0 = \frac{1}{2}; \\ 2(\gamma_1 + l_1\hat{\gamma}_1 + l_0\hat{\gamma}_0) + \beta_1 + l_1^2\hat{\beta}_1 + l_0^2\hat{\beta}_0 = \frac{1}{3}; \\ 3(\gamma_1 + l_1^2\hat{\gamma}_1 + l_0^2\hat{\gamma}_0) + \beta_1 + l_1^3\hat{\beta}_1 + l_0^3\hat{\beta}_0 = \frac{1}{4}; \\ 4(\gamma_1 + l_1^3\hat{\gamma}_1 + l_0^3\hat{\gamma}_0) + \beta_1 + l_1^4\hat{\beta}_1 + l_0^4\hat{\beta}_0 = \frac{1}{5}; \\ 5(\gamma_1 + l_1^4\hat{\gamma}_1 + l_0^4\hat{\gamma}_0) + \beta_1 + l_1^5\hat{\beta}_1 + l_0^5\hat{\beta}_0 = \frac{1}{6}; \\ 6(\gamma_1 + l_1^5\hat{\gamma}_1 + l_0^5\hat{\gamma}_0) + \beta_1 + l_1^6\hat{\beta}_1 + l_0^6\hat{\beta}_0 = \frac{1}{7}; \\ 7(\gamma_1 + l_1^6\hat{\gamma}_1 + l_0^6\hat{\gamma}_0) + \beta_1 + l_1^7\hat{\beta}_1 + l_0^7\hat{\beta}_0 = \frac{1}{8}; \\ 8(\gamma_1 + l_1^7\hat{\gamma}_1 + l_0^7\hat{\gamma}_0) + \beta_1 + l_1^8\hat{\beta}_1 + l_0^8\hat{\beta}_0 = \frac{1}{9}; \\ 9(\gamma_1 + l_1^8\hat{\gamma}_1 + l_0^8\hat{\gamma}_0) + \beta_1 + l_1^9\hat{\beta}_1 + l_0^9\hat{\beta}_0 = \frac{1}{10}. \end{gathered} \tag{32}$$

By solving the resulting system, one can construct different methods with the degree $p \le 10$. For example let us consider to the following methods:

$$\begin{aligned}
\beta_0 &= 0,2986558517\ 827264 \\
\hat{\gamma}_1 &= -0.0000031226\ 45667 \\
\hat{\gamma}_0 &= 0.0299017683\ 803988 \\
\hat{\beta}_0 &= 0.7013441477903306 \\
l_1 &= 0,8247730474\ 58192 \\
l_0 &= 0,6020752452\ 102308
\end{aligned} \tag{33}$$

$$\begin{aligned}
\beta_0 &= 0,4277717272\ 166551 \\
\hat{\gamma}_0 &= 0,0439372763\ 7262437 \\
\gamma_0 &= 0.0610543976\ 1914382 \\
\hat{\beta}_0 &= 0,5722282727833449 \\
l_1 &= 0,6449489742\ 783179 \\
l_0 &= 0,8438640691\ 497394
\end{aligned} \tag{34}$$

$$\begin{aligned}
\beta_1 &= 0,0581973630\ 4284103 \\
\beta_0 &= 0,1919017245\ 988311 \\
\gamma_0 &= 0.0114988044\ 2545037 \\
\hat{\beta}_1 &= 0.3222291309\ 798016 \\
\hat{\beta}_0 &= 0,4276717813784907 \\
l_1 &= 0,7972984557\ 901002 \\
l_0 &= 0,4054301722\ 504762
\end{aligned} \tag{35}$$

$$\begin{aligned}
\beta_1 &= 0,0359591269\ 7178637 \\
\beta_0 &= 0,1399856910\ 166386 \\
\hat{\gamma}_1 &= -0.0508597969\ 4583521 \\
\hat{\gamma}_0 &= -0.0001791537\ 505939 \\
\gamma_1 &= 0.0002705193\ 9103296 \\
\gamma_0 &= 0.0065536104\ 151689 \\
\hat{\beta}_1 &= 0.4937560659\ 942499 \\
\hat{\beta}_0 &= 0.3302991160 1732503 \\
l_1 &= 0,7112214657\ 593782 \\
l_0 &= 0,2992450438\ 9157963
\end{aligned} \tag{36}$$

$$\beta_1 = 0,1225746666\ 3242104$$

$$\begin{aligned}
\beta_0 &= 0{,}1569284773\ \ 7929064 \\
\hat{\gamma}_1 &= -0.0044005543\ \ 697275 \\
\hat{\gamma}_0 &= -0.0000072036\ \ 5772375 \\
\gamma_1 &= 0.0043736622\ \ 9708566 \\
\gamma_0 &= 0.0075140621\ \ 7562786 \\
\hat{\beta}_1 &= 0.3408879308\ \ 97861 \\
\hat{\beta}_0 &= 0.37960892508921557 \\
l_1 &= 0{,}7294647883\ \ 404968 \\
l_0 &= 0{,}3425286935\ \ 7746973
\end{aligned} \tag{37}$$

Consider the application of the method (36) and (37) to solving of the following equation:

$$1.\ y(x) = 1 + \frac{x^2}{2} + \int_0^x e^{x-s}(1 + \frac{s^2}{2})ds, \tag{38}$$

Exact solution for which it the following:

$$y(x) = \frac{x^3}{6} - x - 1;$$

where the user decides to move $h = 0{,}1$. The results can be found in Table 1.

| *Number of example* | $x$ | *Method (36)* | *Method (37)* |
|---|---|---|---|
| $h = 0{,}1$ | 010 | 0 | 0.11E-09 |
| | 0.40 | 0.2E-15 | 0.53E-09 |
| | 0.70 | 0.4E-15 | 0.11E-08 |
| | 1.00 | 0.13E-14 | 0.18E-08 |

In the solving integral equation of (38) have used the algorithm from the [31].

## 5. CONCLUSIONS

Considering that the solution of ordinary differential equations is fundamentally investigated. Using the examples above, we have shown some of the advantages of hybrid methods. As is known the order of accuracy of the method of Simpson coincides with the above given order of accuracy of the hybrid method, but the comparison of the results set forth in these tables routinely comes that the stability region for the hybrid method is more extended. Several authors by using the advantages of hybrid methods applied them to the solving of some problems in mechanics and called them the fractional steps size methods.

It is not difficult to understand that the use of hybrid methods to the solving of the equation (1) gives better results than the corresponding known methods (see e.g. [8]-[10]).

Consequently, the use of hybrid methods in the scientific literature for 60 years shows that these methods are more promising. From the foregoing, it follows that in the theory of hybrid methods, there are many unsolved problems.

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## REFERENCES

[1] E.M. Polishuk Vito Volterra. Leningrad, Nauka, 1977, 114p.
[2] V. Volterra Theory of functional and of integral and integro-differensial equations, Dover publications. Ing, New York, Nauka, Moscow, 1982 p.304 (in Russian).
[3] A.V. Manzhirov, A.D. Polyanin Handbook of Integral Equations: Methods of solutions. Moscow: Publishing House of the "Factorial Press", 2000, 384 p.
[4] A.F. Verlan, V.S. Sizikov. Integral equations: methods, algorithms, programs. Kiev, Naukovo Dumka, 1986, 384 p.
[5] Y.D Mamedov. Methods of calculation, 1978.
[6] A.Quarteroni, R. Sacco, F. Saleri. Numerical Mathematics, Second Edition, Springer, 656p.
[7] R.L.Burden, J.D.Faires Numerical analysis. Cengege Lerning, 2001, 850 pp, 7-th edition.
[8] G.Yu. Mehdiyeva, M.N. Imanova, V.R. Ibrahimov On one application of forward jumping methods. Applied Numerical Mathematics, Volume 72, October 2013, p. 234–245.
[9] P.Linz Linear Multistep methods for Volterra Integro-Differential equations, Journal of the Association for Computing Machinery, Vol.16, No.2, April 1969, p. 295-301.
[10] G.Yu Mehdiyeva, M.N. Imanova, V.R. Ibrahimov Application of the hybrid method with constant coefficients to solving the integro-differential equations of first order. 9th International conference on mathematical problems in engineering, aerospace and sciences, AIP, Vienna, Austria, 10-14 July 2012, p. 506-510.
[11] G.Yu Mehdiyeva, M.N. Imanova, V.R. Ibrahimov On a Research of Hybrid Methods. Numerical Analysis and Its Applications, Springer, 2013, p. 395-402.
[12] V.R.Ibrahimov V.D.Aliyeva The construction of the finite-difference method and application Proceedings of the International Conference on Numerical Analysis and Applied Mathematics 2014 (ICNAAM-2014) AIP Conf. Proc. 1648, 850049-1–850049-5.
[13] A.A.Makroglou Block - by-block method for the numerical solution of Volterra delay integro-differential equations, Computing 3, 1983, 30, №1, p.49-62.
[14] A.Feldstein, J.R. Sopka Numerical methods for nonlinear Volterra integro-differential equations. SIAM J. Numer. Anal. 1974, V 11, p. 826-846.
[15] V.R. Ibrahimov, M.N. Imanova On a Research of Symmetric Equations of Volterra Type. International journal of mathematical models and methods in Applied sciences Volume 8, 2014, p.434-440.
[16] M.N. Imanova One the multistep method of numerical solution for the Volterra integral equation. Transactions issue mathematics and mechanics series of physical -technical and mathematical science, XXBI, 2006, №1.
[17] A.Makroglou Hybrid methods in the numerical solution of Volterra integro-differential equations. Journal of Numerical Analysis 2, 1982, p.21-35.
[18] V.R.Ibrahimov V.D.Aliyeva The construction of the finite-difference method and application Proceedings of the International Conference on Numerical Analysis and Applied Mathematics 2014 (ICNAAM-2014) AIP Conf. Proc. 1648, 850049-1–850049-5.
[19] E. Hairier, S.P.Norsett, G.Wanner Solving ordinary differential equations. (Russian) M., Mir, 1990, 512 p.
[20] Modern numerical methods for solving ordinary differential equations. Editors J.Holl and J.Watt Publishing House "Mir", Moscow, 1979.
[21] G.Dahlquist Convergence and stability in the numerical integration of ordinary differential equations. Math. Scand. 1956, №4, p.33-53.
[22] G. Dahlquist, Stability and Error Bounds in the Numerical Integration of Ordinary Differential Equations. Trans. Of the Royal Inst. Of Techn. Stockholm, Sweden, 1959, №130, p.3-87.
[23] G.Yu Mehdiyeva, V.R. Ibrahimov On the research of multistep methods with constant coefficients, LAP LAMBERT Academic Publishing, 2013, 314 p. (Russian).
[24] G. Yu. Mehdiyeva, M. N. Imanova, V. R. Ibrahimov An application of the hybrid method of multistep type. Advances in Applied and Pure mathematics, Proceedings of 2 Intern.Conf. on Math.Comp and Aqtatist. Science (MCSS), 2014, p. 270-276.
[25] G.K. Gupta. A polynomial representation of hybrid methods for solving ordinary differential equations.Mathematics of comp., volume 33, number 148, 1979, p.1251-1256.
[26] L.M. Skvortsov Explicit two-step Runge-Kutta methods. Math. modeling, 21, 2009, №9, p. 54-65.
[27] G.Yu. Mehdiyeva, M.N. Imanova, V.R. Ibrahimov Some application of the hybrid methods to solving Volterra integral equations Advances in Applied and Pure mathematics, Proceedings of 2 Intern.Conf. on Math.Comp and Aqtatist.Science (MCSS), 2014, p. 352-356.
[28] E.A.Areo, R.A. Ademiluyi, P.O. Babatola Accurate collocation multistep method for integration of first order ordinary differential equations. J.of Modern Math.and Statistics, 2(1): 1-6, 2008, p.1-6.

[29] V. Ibrahimov On the maximal degree of the k-step Obrechkoff's method. Bulletin of Iranian Mathematical Society, Vol.28, №1, 2002, p. 1-28.
[30] V.R. Ibrahimov On a relation between order and degree for stable forward jumping formula. Zh.Vychis.Mat., № 7, 1990, p.1045-1056.
[31] G.Yu. Mehdiyeva, V.R. Ibrahimov, M.N. Imanova The Application Difference Methods to Solving Volterra Integral Equation Pensee Journal, Paris, Vol. 75, Issue. 111, 2013, p. 393-400.