A HIGHER ORDER RATIONAL DIFFERENCE EQUATIONS ON THE DYNAMICS

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ABSTRACT *In this paper we discussed about on the dynamics of a higher order rational difference equations. The global stability of the positive solutions and the periodic character of the difference equation*

$$X_{n+1} = pX_n + qX_{n-t} + rX_{n-l} + \frac{sX_{n-k}+tX_{n-s}}{\alpha X_{n-k}+\beta X_{n-s}} \quad , n = 1,2, \ldots$$

With positive parameter and non-negative initial conditions.

KEYWORDS

Difference equations, Stability, Global stability, Boundedness, Periodic solutions.

1.INTRODUCTION:

Difference equations, although their forms look very simple, it is extremely difficult to understand thoroughly the periodic character, the boundedness character and the global behaviors of their solutions. The study of non-linear rational difference equations of higher order is of paramount importance, since we still know so little about such equations.

In recent years non-linear difference equations have attracted the interest of many researchers, for example:

Kalabusic et al. investigated the periodic nature, boundedness character, and the global asymptotic stability of solutions of the difference

$$Y_{n+1} = \alpha_n + \frac{y_{n-1}}{y_{n-2}}, \text{ n} = 0,1,\ldots.$$

Where the sequence α_n is periodic with period k_2 = {2,3} with positive terms and the initial conditions are positive.

Raafat studied the global attractivity, periodic nature, oscillation and the boundedness of all admissible solutions of the difference equations

$$Y_{n+1} = \frac{A-By_{n-1}}{\pm C+Dy_{n-2}}, \text{ n} = 0,1 ,\ldots\ldots$$

where A,B are non-negative real numbers, C,D are positive real numbers $\pm C + Dy_{n-2} \neq 0$ and for all n ≥ 0.

Alaa investigated the global stability, the permanence , and the oscillation character of the recursive sequence

$$y_{n+1} = \alpha \frac{y_{n-1}}{y_n}, \quad n = 0,1,\ldots$$

where α is a negative number and the initial conditions y_{-1} and y_0 are negative numbers.

Obaid et al. investigated the global stability character, boundedness and the periodicity of solutions of the recursive sequence

$$y_{n+1} = py_n + \frac{qy_{n-1}+ry_{n-2}+sy_{n-3}}{\alpha y_{n-1}+\beta y_{n-2}+\gamma y_{n-3}},$$

where the parameters a,b c,d $\alpha, \beta,$ and γ are positive real numbers and the initial conditions y_{-3}, y_{-2}, y_{-1} *and* y_0 are positive real numbers.

2.PRILIMINARIES

DEFINITION 2.1

A point $\bar{X} \epsilon$ I is called an equilibrium point of the difference equation

$X_{n+1} = F(X_n, X_{n-1}, \ldots X_{n-\delta})$, n=0,1,…. …….. (1)

if $\bar{X} = F(\bar{X}, \bar{X}, \ldots \bar{X})$.

That is, $X_n = \bar{X}$ for n ≥ 0 , is a solution of the difference equation (1), or equivalently, $\bar{X}$ is a fixed point of F.

DEFINITION 2.2

Let $\bar{X} \epsilon (0, \infty)$ be an equilibrium point of the difference equation (1).We have

(i) The equilibrium point $\bar{X}$ of the difference equation(1) is stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $X_{-\delta} \ldots \ldots . . X_{-1} \epsilon$ I with

$$|X_{-\delta} - \bar{X}| +, \ldots |X_{-1} - \bar{X}| + |X_0 - \bar{X}| < \delta,$$

we have

$$|X_n - \bar{X}| < \epsilon \quad \text{for all n} \geq -\delta.$$

(ii) The equilibrium point $\bar{X}$ of the difference equation (1) is called locally asymptotically stable if $\bar{X}$ is called locally stable solution of equation (1) and there exists $\gamma > 0$, such that for all $X_{-\delta} \ldots \ldots . . X_{-1} \epsilon$ I with

$$|X_{-\delta} - \bar{X}| +, \ldots |X_{-1} - \bar{X}| + |X_0 - \bar{X}| < \delta,$$

we have

$$\lim_{n \to \infty} X_n = \bar{X}.$$

(iii) The equilibrium point $\bar{X}$ of the difference equation (1) is called global attractor if for all $X_{-\delta} \ldots \ldots . . X_{-1} \epsilon$ I , we have

$$\lim_{n \to \infty} X_n = \bar{X}.$$

(iv)The equilibrium point $\bar{X}$ of the difference equation (1) is called globally asymptotically stable if $\bar{X}$ is locally stable , and $\bar{X}$ is also a global attractor of the difference equation (1).

(v) The equilibrium point $\bar{X}$ of the difference equation (1) is called unstable if $\bar{X}$ is not locally stable .

DEFINITION 2.3

A sequence $\{X_n\}_{n-\delta}^{\infty}$ is said to be periodic with period p if $Y_{n+p} = Y_n$ for all n $\geq -\delta$.A sequence $\{X_n\}_{n-\delta}^{\infty}$ is said to be periodic with prime period *p* if *p* is the smallest positive integer having this property,

DEFINITION 2.4

Equation (1) is called permanent and bounded if there exists numbers m and M with $0<m<M<\infty$ such that for any initial conditions $X_{-\delta} \dots\dots X_{-1} \epsilon$ (0, ∞) there exist a positive integer N which depends on these initial conditions such that $m \le X_n \le M$ for all n > N.

DEFINITION 2.5

The linearized equation of the difference equation (1) about the equilibrium $\bar{X}$ is the linear difference equation.

$$Y_{n+1} = \sum_{i=0}^{\delta} \frac{\partial F(\bar{X},\bar{X},\dots\bar{X})}{\partial x_{n-i}} Y_{n-i} \qquad \dots\dots\dots\dots(2)$$

Now, assume that the characteristic equation associated with is

$$p(\lambda) = p_0 \lambda^{\delta} + p_1 \lambda^{\delta-1} + \cdots p_{\delta-1}\lambda + p_{\delta} = 0$$

where

$$p_i = \frac{\partial F(\bar{X},\bar{X},\dots\bar{X})}{\partial x_{n-i}}.$$

DEFINITION 2.6

Differential equation is a type of equation which contains derivatives of an unknown function. A higher order differential equation is an equation containing dependent and independent variables and two or more derivatives of the dependent variable with respect to one or more independent variable.

3.LOCAL STABILITY

THEOREM 3.1

Assume that d$\beta \neq \alpha$e, 1 > a + b + c and 2$|(d\beta - \alpha e)|$ < $(\alpha + \beta)$(d+e) ………(3)

then the equilibrium point $\bar{X}$ of equation

$$X_{n+1} = pX_n + qX_{n-1} + rX_{n-1} + \frac{sX_{n-k} + tX_{n-s}}{\alpha X_{n-k} + \beta X_{n-s}}, \text{ n=0,1,2,..} \quad \dots.(A)$$

is locally asymptotically stable.

PROOF:

Let $\frac{(\alpha e - d\beta)(1-a-b-c)}{(\alpha+\beta)(d+e)} = p_5$ ………(4) and 2$|(d\beta - \alpha e)|$ < $(\alpha + \beta)$(d+e)

we deduce that

$$|p_1| + |p_2| + |p_3| + |p_4| + |p_5| < 1$$

$$|a| + |b| + |c| + \left|\frac{(d\beta - \alpha e)(1-a-b-c)}{(\alpha+\beta)(d+e)}\right| + \left|\frac{(\alpha e - d\beta)(1-a-b-c)}{(\alpha+\beta)(d+e)}\right| < 1$$

$$\frac{2(1-a-b-c)}{(\alpha+\beta)(d+e)} |(d\beta - \alpha e)| < 1 - a - b - c$$

If 1 – a – b – c , then

2$|(d\beta - \alpha e)|$ < $(\alpha + \beta)$(d+e).

4.GLOBAL STABILITY:

THEOREM 4.1

The equilibrium point $\bar{X}$ is a global attractor of equation (A) if one of the following conditions holds:

(i) $d\beta - \alpha e$ > 0 , a + b + c < 1

(ii) $\alpha e - d\beta$ > 0 , a + b + c < 1.

PROOF:

Let r and s be non-negative real numbers and assume that g : $[r, s]^5 \rightarrow [r, s]$ be a function defined by g(

$v_0, v_1, v_2, v_3, v_4) = av_0 + bv_1 + cv_2 + \frac{dv_3 + ev_4}{\alpha v_3 + \beta v_4}$

Then

$$\frac{\partial g(v_0, v_1, v_2, v_3, v_4)}{\partial v_0} = a\, , \frac{\partial g(v_0, v_1, v_2, v_3, v_4)}{\partial v_1} = b\, , \frac{\partial g(v_0, v_1, v_2, v_3, v_4)}{\partial v_2} = c,$$

$$\frac{\partial g(v_0, v_1, v_2, v_3, v_4)}{\partial v_3} = \frac{(d\beta - \alpha e)v_4}{(\alpha v_3 + \beta v_4)^2}, \quad \frac{\partial g(v_0, v_1, v_2, v_3, v_4)}{\partial v_4} = \frac{(\alpha e - d\beta)v_3}{(\alpha v_3 + \beta v_4)^2}$$

we consider two cases:

Case 1:

Let $d\beta - \alpha e > 0$, a + b + c < 1 and $e - d \neq 0$ is true, then we can easily see that the function g($v_0, v_1, v_2, v_3, v_4)$ is increasing in v_0, v_1, v_2, v_3 and decreasing in v_4 .suppose that (m , M) is a solution of the system M=h(M , M , M , m) and m=h(m , m , m , M).

Then from equation , we see that

M = aM+bM+cM+$\frac{dM+em}{\alpha M+\beta m}$ and m = am+bm+Cm+$\frac{dm+eM}{\alpha m+\beta M}$

then

$\alpha(1 - a - b - c)M^2 + \beta(1 - a - b - c)$mM = dM+em,

$\alpha(1 - a - b - c)m^2 + \beta(1 - a - b - c)$Mm = dm+eM,

Subtracting this two equations , we obtain

(M – m)[$\alpha(1 - a - b - c)(M + m) + (e - d)$] = 0,

under the condition a+ b+ c $\neq 1$ and e $\neq$ d we see that M = m. It follows from the Theorem " Let g : [η, ξ] is an interval of real numbers. Consider the difference equation $X_{n+1} = g(X_n, X_{n-1}, \ldots X_{n-\delta})$, n = 0,1,….." that $\bar{X}$ is a global attractor of equation (A).

Case 2

Let $\alpha e - d\beta > 0$, a + b + c < 1, and $(\alpha - \beta) > 0$, a + b + c < 1$\neq$ d − e is true, then we can say easily see that the function g($v_0, v_1, v_2, v_3, v_4)$ is increasing in v_0, v_1, v_2, v_4 and decreasing in v_3. Suppose that (m , M) is a solution of the system

M = aM+bM+cM+$\frac{dm+eM}{\alpha m+\beta M}$ and m = am+bm+Cm+$\frac{dM+em}{\alpha M+\beta m}$

then

$\alpha(1 - a - b - c)$Mm $+\alpha(1 - a - b - c)M^2$ = dm +eM,

$\alpha(1 - a - b - c)$mM $+ \alpha(1 - a - b - c)m^2$ = dM+em,

Subtracting this two equations, we obtain

(M − m) [β(1− a − b − c)(M + m)+(d − e)] = 0.

under the conditions a+b+c $\neq$ 1 and e $\neq$ d we see that M = m. It follows from the Theorem " Let g : [η, ξ] is an interval of real numbers. Consider the difference equation

$X_{n+1} = g(X_n, X_{n-1}, \ldots X_{n-\delta})$, n = 0,1,….."

that $\bar{X}$ is a global attractor of equation (A).

THEOREM 4.2

Every solution of equation (A) is bounded if a + b + c < 1.

PROOF:

Let $\{X_n\}_{n=-\delta}^{\infty}$ be a solution of equation (A).It follows from equation (A) that

$$X_{n+1} = pX_n + qX_{n-t} + rX_{n-l} + \frac{sX_{n-k} + tX_{n-s}}{\alpha X_{n-k} + \beta X_{n-s}},$$

$$X_{n+1} = pX_n + qX_{n-t} + rX_{n-l} + \frac{sX_{n-k}}{\alpha X_{n-k} + \beta X_{n-s}} + \frac{tX_{n-s}}{\alpha X_{n-k} + \beta X_{n-s}}$$

$$\leq pX_n + qX_{n-t} + rX_{n-l} + \frac{sX_{n-k}}{\alpha X_{n-k}} + \frac{tX_{n-s}}{\beta X_{n-s}}$$

$$\leq pX_n + qX_{n-t} + rX_{n-l} + \frac{d}{\alpha} + \frac{e}{\beta}.$$

we have $X_{n-t} \le Z_{n+1}$, where $Z_{n+1} = pZ_n + qZ_{n-t} + rZ_{n-l} + \frac{d}{\alpha} + \frac{e}{\beta}$ linear non-homogeneous equation. It is easy to see that the solution of this equation is locally asymptotically stable and converges to the equilibrium point $\bar{Z} = \frac{d\beta + \alpha e}{\alpha\beta[1-(a+b+c)]}$ if a +b + c <1 . By using the inequality theorem we have that the solution of (A) is bounded.

THEOREM 4.3
Every solution of equation (A) is unbounded if a > 1 or b > 1 or c > 1.

PROOF:
Let $\{X_n\}_{n=-\delta}^{\infty}$ be a solution of equation (A). Then from equation (A) we see that
$X_{n+1} = pX_n + qX_{n-t} + rX_{n-l} + \frac{SX_{n-k}+tX_{n-s}}{\alpha X_{n-k}+\beta X_{n-s}} > pX_n$ for all n ≥ 1.
We see that the right hand side can be written as follows
$X_{n+1} = pX_n$.
Then
$Z_n = p^n Z_0$
and this equation is unstable because a > 1, and $\lim_{n\to\infty} Z_n = \infty$.Then by ratio test $\{X_n\}_{n=-\delta}^{\infty}$ is bounded from above. Using the same technique, we can prove the other cases.

5. PERIODIC SOLUTIONS:

THEOREM 5.1
If t ,l ,k , are an even and s is an odd then equation (A) has a prime period two solutions if and only if
(e−d) $(\alpha - \beta)$(a+b+c+1)$-4(e\alpha(a+b+c) + d\beta)$ > 0…..(4)

PROOF:
First suppose that there exists a prime period two solution ….p ,q ,p, q,…, of equation (A).If t ,l and k are even and s is an odd then $X_n = X_{n-t} = X_{n-l} = X_{n+1}$ and $X_{n+1} = X_{n-s}$. It follows from equation (A) that
p=aq+bq+cq+$\frac{dq+ep}{\alpha p+\beta q}$.
Therefore,
$\beta p^2 + \alpha pq = \alpha(a+b+c)q^2 + \beta(a+b+c)pq + dq + ep$,…(5) and
$\beta q^2 + \alpha pq = \alpha(a+b+c)p^2 + \beta(a+b+c)pq + dp + eq$,…(6)
By subtracting (6) from (5),we deduce
p+ q = $\frac{e-d}{\beta+\alpha(a+b+c)}$ ……………..(7)
Again, adding (5) and (6), we have
p q = $\left(\frac{(e-d)(e\alpha(a+b+c)+d\beta}{(\beta+\alpha(a+b+c))^2}\right)\left(\frac{1}{(\alpha-\beta)(a+b+c+1)}\right)$ ………(8)
Where $e > d$ and $\alpha > \beta$.
Let p and q are the two positive distinct real roots of the quadratic equation
$t^2 -$ (p+ q)t+ p q = 0,
$t^2 - \left(\frac{e-d}{\beta+\alpha(a+b+c)}\right)t + \left(\frac{(e-d)(e\alpha(a+b+c)+d\beta}{(\beta+\alpha(a+b+c))^2(\alpha-\beta(a+b+c+1)}\right) = 0$ …….(9)
Thus we deduce
$\left(\frac{(e-d)}{\beta+\alpha(a+b+c)}\right)^2 - 4\left(\frac{(e-d)(e\alpha(a+b+c)+d\beta}{(\beta+\alpha(a+b+c))^2}\right) \times \left(\frac{1}{(\alpha-\beta)(a+b+c+1)}\right) > 0,$ or
(e−$d)(\alpha-\beta)(a+b+c+1) - 4(e\alpha(a+b+c+1)+d\beta) > 0$
Therefore Inequality (4) holds.
Second suppose that inequality (4) is true.
We will show that equation (A) has a prime period two solution.
Suppose that

p = $\frac{(e-d)+\zeta}{2(\beta+\alpha A)}$ and q = $\frac{(e-d)+\zeta}{2(\beta+\alpha A)}$

where

$\zeta = \sqrt{(e-d)^2 - \frac{4(e-d)(e\alpha A+d\beta)}{(\alpha-\beta)(A+1)}}$ and A=a +b +c.

Therefore p and q are distinct real numbers.

Set $Y_{-t} = q, Y_{-l} = q, Y_{-K} = q, y_{-s} = p, \ldots Y_{-3} = p, Y_{-2} = q, Y_{-1} = p, Y_{0}$, we would like to show that $Y_1 = Y_{-1} = p$ and $Y_2 = Y_0 = q$. It follows from equation (A) that

$$Y_1 = aq + bq + cq + \frac{dq+ep}{\alpha q+\beta p} = Aq + \frac{d\left(\frac{(e-d)-\zeta}{2(\beta+\alpha A)}\right)+e\left(\frac{(e-d)+\zeta}{2(\beta+\alpha A)}\right)}{\alpha\left(\frac{(e-d)-\zeta}{2(\beta+\alpha A)}\right)+\beta\left(\frac{(e-d)+\zeta}{2(\beta+\alpha A)}\right)}$$

Dividing the denominator and numerator by $2(\beta+\alpha A)$ we get, $Y_1 = Aq +$

$$\frac{d((e-d)-\zeta)+e((e-d)+\zeta)}{\alpha((e-d)-\zeta)+\beta((e-d)+\zeta)} = Aq +$$

$$\frac{(e+d)(e-d)+e((e-d)+\zeta)}{(\alpha+\beta)(e-d)+(\beta-\alpha)\zeta}$$

Multiplying the denominator and numerator of the right side by

$(\alpha+\beta)(e-d)-(\beta-\alpha)\zeta$ $Y_1 = Aq +$

$$\frac{d((e-d)-\zeta)+e((e-d)+\zeta)}{\alpha((e-d)-\zeta)+\beta((e-d)+\zeta)}$$

$$= Aq + \frac{(e+d)(e-d)+(e-d)\zeta}{(\alpha+\beta)(e-d)+(\beta-\alpha)\zeta}.$$

Multiplying the denominator and numerator of the right side by $(\alpha+\beta)(e-d)+(\beta-\alpha)\zeta$

$$= Aq + \frac{(e-d)[(e+d)+\zeta][(\alpha+\beta)(e-d)+(\beta-\alpha)\zeta]}{[(\alpha+\beta)(e-d)+(\beta-\alpha)\zeta][(\alpha+\beta)(e-d)+(\beta-\alpha)\zeta]}.$$

$$= Aq + \frac{(e-d)[(e+d)(\alpha+\beta)(e-d)+2(\alpha e-\beta d)\zeta-(\beta-\alpha)\zeta^2]}{(\alpha+\beta)^2(e-d)^2-(\beta-\alpha)^2\zeta^2}. \quad = Aq+$$

$$\frac{(e-d)\left[(e+d)(\alpha+\beta)(e-d)+2(\alpha e-\beta d)\zeta-(\beta-\alpha)\left((e-d)^2-\frac{4(e-d)(e\alpha A+d\beta)}{(\alpha-\beta)(A+1)}\right)\right]}{(\alpha+\beta)^2(e-d)^2-(\beta-\alpha)^2\left((e-d)^2-\frac{4(e-d)(e\alpha A+d\beta)}{(\alpha-\beta)(A+1)}\right)}.$$

$$= Aq + \frac{(e-d)\left[2(e-d)(\alpha e-\beta d)+2(\alpha e-\beta d)\zeta-\left(\frac{4(e-d)(e\alpha A+d\beta)}{(A+1)}\right)\right]}{4\alpha\beta(e-d)^2\left(\frac{4(e-d)(e\alpha A+d\beta)}{(A+1)}\right)}$$

$$= Aq\,\frac{2(e-d)[(\alpha e-\beta d)(A+1)-2(e\alpha A-d\beta)]+2(A+1)(\alpha e-\beta d)\zeta}{4\alpha\beta(e-d)(A+1)+4(\alpha-\beta)(e\alpha A-d\beta)}.$$

$$= A\left(\frac{(e-d)-\zeta}{2(\beta+\alpha A)}\right) + \frac{(e-d)(\alpha e-\beta d)(1-A)+(A+1)(e-\beta d)\zeta}{2(\alpha A+\beta)(e\alpha-d\beta)}.$$

$$= \frac{(e-d)A-A\zeta+(e-d)(1-A)+(A+1)\zeta}{2(\alpha A+\beta)}.$$

$$= \frac{(e-d)+\zeta}{2(\alpha A+\beta)}.$$

= p

Similarly as before , it is easy to show that $X_2 = q$.

Then by induction we get $X_{2n} = q$ and $X_{2n+1} = p$ for all n $\geq -\delta$.

Thus equation (A) has the prime period two solution ...p, q , p, q ,... where p and q are the distinct roots of the quadratic equation (9) and the proof is complete.

THEOREM 5.2

If t, l ,k , and s are an even and a+ b+ c =$\frac{d+e}{\alpha+\beta} \neq 1$.

Then equation (A) has no prime period two solutions.

PROOF:

Suppose that there exists a prime period two solution …p ,q ,p ,q ,… of equation (A). we see from equation (A) when t ,l ,k , and s are an

$$\left(1-a-b-c-\frac{d+e}{\alpha+\beta}\right)(p-q) = 0$$

since

$1 - a - b - c - \frac{d+e}{\alpha+\beta} \neq 1$, then p = q.

This is a contradiction.

6.CONCLUSION:

We have discussed about the on the dynamics of a higher order rational difference equations. The global stability of the positive solutions and the periodic character of the difference equation

$X_{n+1} = pX_n + qX_{n-t} + rX_{n-l} + \frac{sX_{n-k}+tX_{n-s}}{\alpha X_{n-k}+\beta X_{n-s}}$, n=1,2,…..

With positive parameter and non-negative initial conditions.

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