A STUDY OF A SPECIAL THEORY OF TWO POINT LINEAR DIFFERENTIAL OPERATORS ON FINITE INTERVAL

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ABSTRACT

In this paper we will discuss a study of a special theory of two point linear differential operators on finite interval. This paper will be based on the theory of linear two point initial boundary values problem, the spectral theory of linear deferential operators and the connections between two fields. The primary interest in this work is not second order partially differential equation, such as the heat equation but third and higher odd order equations. Spectral Theory of ordinary and partially linear differential operators on finite interval, the initial problems for linear evolution equation with constant coefficient of any order is discussed.

Keyword: - *Fields, Linear, Operators, Boundary values, Spectral Theory etc.*

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1. INTRODUCTION

The primary interest in this work is not second order partially differential equation, such as the heat equation, third and high odd order equation. For the case of second order partially differential equation these equations are fully resolved at least when the solution and the data both satisfy some differentiability condition. Indeed, Cauchy not only posed the problem but solved it for analytic data. Hadamard examined question in particular for second order problem. However when the partially differential equation is of a higher order the application of their techniques works only with very specific types of boundary conditions. Special attention is paid to the question of separated boundary conditions, spectral multiplicity and absolutely continuous spectrum. For the case nm=2 (Sturm-Liouville operators and Dirac systems) the classical theory of Weyl-Titchmarch is included. Oscillation theory for Sturm-Liouville operators and Dirac systems is developed and applied to the study of the essential and absolutely continuous spectrum. The results are illustrated by the explicit solution of a number of particular problems including the spectral theory one partial Schrödinger and Dirac operators with spherically symmetric potentials. The methods of proof are functionally analytic wherever possible.

2. BOUNDARY CONDITIONS

- If each boundary condition has involves only a single order of spatial derivative (though possibly at both ends) then we call the boundary conditions non-Robin. Boundary conditions are non-Robin if each contains only one order of partial derivative. Otherwise we say that boundary condition is of Robin type.

- Boundary conditions with the property every non-zero entry in the boundary coefficient matrix is a pivot are called simple.

Example: The boundary conditions

$q_x(0,t)= q_x(1,t)$

$q(0,t)= q(1,t)=0$

may be expressed by specifying the boundary data h1 = h2 = h3 = 0 and boundary coefficient matrix

$$A = \begin{pmatrix} 0\,0\,1 & -1\,0\,0 \\ 0\,0\,0 & 0\,1\,0 \\ 0\,0\,0 & 0\,0\,1 \end{pmatrix}$$

Hence these boundary conditions are homogeneous and non-Robin but coupled.

3. SPECTRAL THEORY OF DIFFERENTIAL OPERATORS

The branch of the general spectral theory of operators in which one investigates the spectral properties of differential operators on various function spaces, especially on Hilbert spaces of measurable functions.

Let Ω_n be a domain in $\mathbf{R}^n$, let Γ be its boundary, let

$$l(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha \qquad (1)$$

be a linear differential operator, and let

$$l_j(u) = \sum_{|\alpha| \leq m_1} b_{\alpha,j}(x) D^\alpha u|_\Gamma = 0, \qquad 1 \leq j \leq N, \qquad (2)$$

be the boundary conditions, defined by linear differential operators l_j.

Here

$$x = (x_1, \dots, x_n), \qquad D = (D_1, \dots, D_n), \qquad D_j = \frac{\partial}{\partial x_j},$$

$$\alpha = (\alpha_1, \dots, \alpha_n),$$

the α_j are non-negative integers, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, and a_α and $b_{\alpha,j}$ are functions defined in Ω_n and on Γ, respectively. Unless otherwise stated, in the sequel it is assumed that a_α and $b_{\alpha,j}$ are sufficiently smooth functions when $n > 1$, and that $a_m(x) \neq 0$ for all $x \in (a, b)$, where $\Omega_1 = (a, b)$ if $n = 1$.

4. SPECTRAL THEORY

Spectral theory for second order ordinary differential equations on a compact interval was developed by Jacques Charles François Sturm and Joseph Liouville in the nineteenth century and is now known as Sturm–Liouville theory. In modern language it is an application of the spectral theorem for compact operators due to David Hilbert. In his paper, published in 1910, Hermann Weyl extended this theory to second order ordinary differential equations with singularities at the endpoints of the interval, now allowed to be infinite or semi-infinite. He simultaneously developed a spectral theory adapted to these special operators and introduced boundary conditions in terms of his celebrated dichotomy between *limit points* and *limit circles*.

In the 1920s John von Neumann established a general spectral theorem for unbounded self-adjoint operators, which Kunihiko Kodaira used to streamline Weyl's method. Kodaira also generalised Weyl's method to singular ordinary differential equations of even order and obtained a simple formula for the spectral measure. The same formula had also been obtained independently by E. C. Titchmarsh in 1946 (scientific communication between Japan and the United Kingdom had been interrupted by World War II). Titchmarsh had followed the method of the German mathematician Emil Hilb, who derived the eigenfunction expansions using complex function theory instead of operator theory. Other methods avoiding the spectral theorem were later developed independently by Levitan, Levinson and Yoshida, who used the fact that the resolvent of the singular differential operator could be approximated by compact resolvents corresponding to Sturm–Liouville problems for proper subintervals. Another method was found by Mark Grigoryevich Krein; his use of *direction functionals* was subsequently generalised by Izrail Glazman to arbitrary ordinary differential equations of even order.

Weyl applied his theory to Carl Friedrich Gauss's hypergeometric differential equation, thus obtaining a far-reaching generalisation of the transform formula of Gustav Ferdinand Mehler (1881) for the Legendre differential equation, rediscovered by the Russian physicist Vladimir Fock in 1943, and usually called the Mehler–Fock transform. The corresponding ordinary differential operator is the radial part of the Laplacian operator on 2-dimensional hyperbolic space. More generally, the Plancherel theorem for SL(2,R) of Harish Chandraand Gelfand–Naimark can be deduced from Weyl's theory for the hypergeometric equation, as can the theory of spherical functions for the isometry groups of higher dimensional hyperbolic spaces. Harish Chandra's later development of the Plancherel theorem for general real semisimple Lie groups was strongly influenced by the methods Weyl developed for eigenfunction expansions associated with singular ordinary differential equations. Equally importantly the theory also laid the mathematical foundations for the analysis of the Schrödinger equation and scattering matrix in quantum mechanics.

5. SPECIAL THEORY OF TWO POINT LINEAR DIFFERENTIAL OPERATORS

The spectral theory of ordinary differential equations is the part of spectral theory concerned with the determination of the spectrum and eigenfunction expansion associated with a linear ordinary differential equation. In his paper Hermann Weyl generalized the classical Sturm–Liouville theory on a finite closed interval to second order differential operators with singularities at the endpoints of the interval, possibly semi-infinite or infinite. Unlike the classical case, the spectrum may no longer consist of just a countable set of eigenvalues, but may also contain a continuous part. In this case the eigenfunction expansion involves an integral over the continuous part with respect to a spectral measure, given by the Titchmarsh–Kodaira formula. The theory was put in its final simplified form for singular differential equations of even degree by Kodaira and others, using von Neumann's spectral theorem. It has had important applications in quantum mechanics, operator theory and harmonic analysis on semisimple Lie groups. Spectral theory for second order ordinary differential equations on a compact interval was developed by Jacques Charles François Sturm and Joseph Liouville in the nineteenth century and is now known as Sturm–Liouville theory. In modern language it is an application of the spectral theorem for compact operators due to David Hilbert. In his paper, published in 1910, Hermann Weyl extended this theory to second order ordinary differential equations with singularities at the endpoints of the interval, now allowed to be infinite or semi-infinite. He simultaneously developed a spectral theory adapted to these special operators and introduced boundary conditions in terms of his celebrated dichotomy between *limit points* and *limit circles*.

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6. CONCLUSION

In this paper we are discussing only some important point of special theory of two point linear differential operators on finite interval. Many authors discussed in this topic. Spectral theory for second order ordinary differential equations on a compact interval was developed by Jacques Charles François Sturm and Joseph Liouville in the nineteenth century and is now known as Sturm–Liouville theory. The primary interest in this work is not second order partially differential equation, such as the heat equation but third and higher odd order equations. The theory was put in its final simplified form for singular differential equations of even degree by Kodaira and others, using von Neumann's spectral theorem. It has had important applications in quantum mechanics, operator theory and harmonic analysis on semisimple Lie groups.

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