A STUDY OF NONLINEAR ORDER DIFFERENTIAL EQUATION USING NDM

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Abstract

In this research paper, we examine a novel method called the Natural Decomposition Method (NDM). We use the NDM to obtain exact solutions for three different types of nonlinear ordinary differential equations (NLODEs). The NDM is based on the Natural transform method (NTM) and the Adomian decomposition method (ADM). By using the new method, we successfully handle some class of nonlinear ordinary differential equations in a simple and elegant way. The proposed method gives exact solutions in the form of a rapid convergence series. Hence, the Natural Decomposition Method (NDM) is an excellent mathematical tool for solving linear and nonlinear differential equation. One can conclude that the NDM is efficient and easy to use.

Keywords: *Natural transform, Sumudu transform, Laplace transform, Adomian decomposition method, ordinary differential equations.*

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1. INTRODUCTION

Nonlinear differential equations have received a considerable amount of interest due to its broad applications. Nonlinear ordinary differential equations play an important role in many branches of applied and pure mathematics and their applications in engineering, applied mechanics, quantum physics, analytical chemistry, astronomy and biology. From last decade, researcher pays attentions towards analytical and numerical solutions of nonlinear ordinary differential equations. Therefore, it becomes increasingly important to be familiar with all traditional and recently developed methods for solving linear and nonlinear ordinary differential equations. We present a new integral transform method called the Natural Decomposition Method (NDM), and apply it to find exact solutions to nonlinear ODEs. There are many integral transform methods exists in the literature to solve ODEs. The most used one is the Laplace transformation. Other methods used recently to solve PDEs and ODEs, such as, the Sumudu transform, the Reduced Differential Transform Method (RDTM) and the Elzaki transform. Fethi Belgacem and R. Silambarasan, used the N–Transform to solve the Maxwell's equation, Bessel's differential equation and linear and nonlinear Klein Gordon Equations and more. Also, Zafar H. Khan and Waqar A. Khan, used the N–Transform to solve linear differential equations and they presented a table with some properties of the N–Transform of different functions.

First, consider the nonlinear second order differential equation of the form:

$$\frac{d^2v}{dt^2} + \left(\frac{dv}{dt}\right)^2 + v^2(t) = 1 - \sin(t), \qquad (1.1)$$

subject to the initial conditions

$$v(0) = 0, \quad v'(0) = 1. \qquad (1.2)$$

Second, the first order nonlinear ordinary differential equation of the form:

$$\frac{dv}{dt} - 1 = v^2(t), \qquad (1.3)$$

subject to the condition

$$v(0) = 0. \qquad (1.4)$$

Third, the nonlinear Riccati differential equation of the form:

$$\frac{dv}{dt} = 1 - t^2 + v^2(t), \qquad (1.5)$$

subject to the condition

$$v(0) = 0. \qquad (1.6)$$

2. FIRST CLASS OF REDUCIBLE SECOND ORDER NDM

In the sequel, we provide with some classes of second order nonlinear differential equations, transformable to first order ones using specific parameter variations. Concrete examples are exhibited as matter of illustration. Specifically, the classes of equations of the forms

$$(y')^m y'' + a(x)(y')^{m+1} = f(x, y, y'), \qquad (1)$$

And,

$$(y')^m y'' + a(y')(y')^{m+2} = f(x, y, y'), \qquad (2)$$

where m is a positive integer and a is an integrable function, are considered. Note that, if the function $f(x, y, y') = g(x, y')$, i.e. the second members of equations (1) and (2) do not explicitly depend on the variable y, then we can perform the natural change of variables $z(x) = y'(x)$ to lower the order of these equations. Besides, if the function a is constant only in equation (1), (not necessarily constant in equation (2)), and if $f(x, y, y') = g(y, y')$, i.e. the equations (1) and (2) are autonomous, then we can perform the change of variables $w(y) = y'(x)$ to transform the above mentioned classes of equations into first order ones.

Let us consider nonlinear second order differential equations of the following type:

$$y'' + a(x)y' = F(x) + y'G(y)e^{-\int a(x)dx} \tag{3}$$

whose the linear part, i.e.

$$y'' + a(x)y' = 0 \tag{4}$$

yields the solution

$$y' = Ce^{-\int a(x)dx}, \tag{5}$$

where C is an arbitrary constant. Suppose that C is a differentiable function of both variables x and y expressed in the form

$$C = H(x) + K(y). \tag{6}$$

Then, (5) can be rewritten as

$$y' = [H(x) + K(y)]\, e^{-\int a(x)dx} \tag{7}$$

that we differentiate to obtain

$$y'' = e^{-\int a(x)dx}\left[H'_x + y'K'_y\right] - a(x)y', \tag{8}$$

where

$$H'_x = \frac{dH(x)}{dx} \quad \text{and} \quad K'_y = \frac{dK(y)}{dy}.$$

Substituting (7) and (8) into (3), we find

$$H'_x + y'K'_y = F(x)e^{\int a(x)dx} + y'G(y). \tag{9}$$

Clearly, (9) will take place if

$$H'_x = F(x)e^{\int a(x)dx} \quad \text{and} \quad K'_y = G(y). \tag{10}$$

We therefore state the following result:

Proposition 1 The second order nonlinear differential equation (3) can be reduced to the first order differential equation (7), where the functions H and K are solutions of the first order differential equations (10), respectively.

As a matter of clarity, let us consider the following example:

Example 1 Let us consider the function G in (3) in the form

$$G(y) = \frac{d}{dy}\left(\frac{b_0 + b_1 y + b_2 y^2 + b_3 y^3}{c_0 + c_1 y}\right), \tag{11}$$

where b0, b1, b2, b3 are arbitrary constants and c0, c1 are constants such that $(c_0, c_1) \neq (0, 0).$ Then equations (10) yield :

$$\begin{aligned} H(x) &= \int F(x) e^{\int a(x)dx} dx \\ K(y) &= A + \frac{b_0 + b_1 y + b_2 y^2 + b_3 y^3}{c_0 + c_1 y}, \end{aligned} \tag{12}$$

where *A* is an arbitrary constant of integration. Therefore, equation (7) is reduced to the well known Abel equation of second kind

$$y' = e^{-\int a(x)dx} \frac{b_0 + c_0(H(x) + A) + [c_1(H(x) + A) + b_1]\, y + b_2 y^2 + b_3 y^3}{c_0 + c_1 y}. \tag{13}$$

Finally, the equation (3) takes the form

$$y'' + a(x)y' = F(x) + y' \frac{d}{dy}\left(\frac{b_0 + b_1 y + b_2 y^2 + b_3 y^3}{c_0 + c_1 y}\right) e^{-\int a(x)dx} \tag{14}$$

and is integrable if the constants b0, b1, b2, b3, c0, c1 and the functions a and *F* are chosen in such a way that the Abel equation (13) be integrable.

In particular, for c0 = 1, c1 = 0 and F = 0, equation (13) leads to the separable Abel equation

$$y' = (A + b_0 + b_1 y + b_2 y^2 + b_3 y^3)\, e^{-\int a(x)dx} \tag{15}$$

whose the implicit solution is given by

$$\int \frac{dy}{A + b_0 + b_1 y + b_2 y^2 + b_3 y^3} = \int e^{-\int a(x)dx} dx + B, \tag{16}$$

where *B* is an arbitrary constant of integration.

It is worth noticing that a second kind Abel equation of the form

$$y' = \frac{f_3 y^3 + f_2 y^2 + f_1 y + f_0}{g_1 y + g_0}, \quad \text{with} \quad f_3 \neq 0, \tag{17}$$

where fi , (i = 0, 1, 2, 3), and gj , (j = 0, 1), are arbitrary functions of x, can be transformed into a canonical form. Indeed, using the variable change

$$\left\{x = t,\ y = \frac{1 - g_0 u}{g_1 u}\right\}, \tag{18}$$

where t and u = u(t) are the new independent and dependent variables, respectively, equation (17) becomes

$$u'_t = \tilde{f}_3 u^3 + \tilde{f}_2 u^2 + \tilde{f}_1 u + \tilde{f}_0. \tag{19}$$

Making use of the substitution

$$u = v - \frac{\tilde{f}_2}{3\,\tilde{f}_3}, \tag{20}$$

equation (19) can be put in the form

$$v'_t = h_3 v^3 + h_1 v + h_0. \tag{21}$$

Now, setting

$$v = E(t)w, \quad \text{where} \quad E(t) = e^{\int h_1(t)dt}, \tag{22}$$

brings this equation to the simpler form:

$$w'_t = \tilde{h}_3 w^3 + \tilde{h}_0, \tag{23}$$

which, in turn, can be reduced, with the help of the new independent variable

$$s = \int \tilde{h}_3(t)dt, \tag{24}$$

to the usual canonical form of Abel equation of the first kind

$$w'_s = w^3(s) + k(s). \tag{25}$$

The latter is integrable by various methods known in the literature. for a good compilation of techniques developed to solve (25) for particular expressions of k(s).

3. DEFINITIONS AND PROPERTIES OF THE N–TRANSFORM

The natural transform of the function f(t) for t ∈ (−∞, ∞) is defined by [11, 12]:

$$\mathbb{N}[f(t)] = R(s,u) = \int_{-\infty}^{\infty} e^{-st} f(ut)\,dt; \quad s,\ u \in (-\infty, \infty), \tag{1}$$

where N [f(t)] is the natural transformation of the time function f(t) and the variables s and u are the natural transform variables. Note that Eq. (1) can be written in the form [4, 5]:

$$\begin{aligned}
\mathbb{N}[f(t)] &= \int_{-\infty}^{\infty} e^{-st} f(ut)\, dt; \quad s,\, u \in (-\infty, \infty) \\
&= \Big[\int_{-\infty}^{0} e^{-st} f(ut)\, dt;\ s,\, u \in (-\infty, 0)\Big] + \Big[\int_{0}^{\infty} e^{-st} f(ut)\, dt;\ s,\, u \in (0, \infty)\Big] \\
&= \mathbb{N}^{-}[f(t)] + \mathbb{N}^{+}[f(t)] \\
&= \mathbb{N}[f(t)H(-t)] + \mathbb{N}[f(t)H(t)] \\
&= R^{-}(s,u) + R^{+}(s,u),
\end{aligned}$$

where H(.) is the Heaviside function.

It should be mentioned here, if the function f(t)H(t) is defined on the positive real axis, with t ∈ R, then we define the Natural transform (N–Transform) on the set

$$A = \left\{ \begin{matrix} f(t) : \exists\, M,\, \tau_1,\, \tau_2 > 0, \text{ such that } |f(t)| < M e^{\frac{|t|}{\tau_j}}, \\ \text{if } t \in (-1)^j \times [0, \infty),\ j \in \mathbb{Z}^{+} \end{matrix} \right\}$$

as:

$$\mathbb{N}[f(t)H(t)] = \mathbb{N}^{+}[f(t)] = R^{+}(s,u) = \int_{0}^{\infty} e^{-st} f(ut)\, dt; \quad s,\, u \in (0, \infty), \tag{2}$$

where H(.) is the Heaviside function. Note if u = 1, then Eq. (2) can be reduced to the Laplace transform and if s = 1, then Eq. (2) can be reduced to the Sumudu transform. Now we give some of the N–Transforms and the conversion to Sumudu and Laplace [11, 12].

Table 1. Special N–Transforms and the conversion to Sumudu and Laplace

$f(t)$	$\mathbb{N}[f(t)]$	$\mathbb{S}[f(t)]$	$\ell[f(t)]$
1	$\frac{1}{s}$	1	$\frac{1}{s}$
t	$\frac{u}{s^2}$	u	$\frac{1}{s^2}$
e^{at}	$\frac{1}{s-au}$	$\frac{1}{1-au}$	$\frac{1}{s-a}$
$\frac{t^{n-1}}{(n-1)!}, n=1,2,...$	$\frac{u^{n-1}}{s^n}$	u^{n-1}	$\frac{1}{s^n}$
$\sin(t)$	$\frac{u}{s^2+u^2}$	$\frac{u}{1+u^2}$	$\frac{1}{1+s^2}$

Remark 1. The reader can read more about the Natural transform in [11, 12].

Now we give some important properties of the N–Transforms are given as follows:

Table 2. Properties of N–Transforms

Functional Form	Natural Transform
$y(t)$	$Y(s,u)$
$y(at)$	$\frac{1}{a}Y(s,u)$
$y'(t)$	$\frac{s}{u}Y(s,u) - \frac{y(0)}{u}$
$y''(t)$	$\frac{s^2}{u^2}Y(s,u) - \frac{s}{u^2}y(0) - \frac{y'(0)}{u}$
$\gamma\, y(t) \pm \beta\, v(t)$	$\gamma\, Y(s,u) \pm \beta\, V(s,u)$

4. CONCLUSION

In this paper, the Natural Decomposition Method (NDM) was proposed for solving the Riccati differential equation and two nonlinear ordinary differential equations. We successfully found exact solutions to all three applications. The NDM introduces a significant improvement in the fields over existing techniques. Our goal in the future is to apply the NDM to other linear nonlinear differential equations (PDEs, ODEs) that arise in other areas of science and engineering.

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