A STUDY OF NUMERICAL METHOD FOR SOLUTION FOR DIFFERENTIAL INTEGRATION EQUATIONS

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ABSTRACT

Differential equations can describe nearly all systems undergoing change. Many mathematicians have studied the nature of these equations for hundreds of years and there are many well-developed solution techniques. Often, systems described by differential equations are so complex, or the systems that they describe are so large, that a purely analytical solution to the equations is not tractable. It is in these complex systems where computer simulations and numerical methods are useful. Traditional methods for solving the boundary value problems of elasticity and thermo elasticity are orientated towards usage of harmonic and biharmonic functions for stresses or displacements. Also, most of the existing methods are aimed at and adjusted to special loadings and forms of the region, and by far do not provide solutions in the form of a functional dependence of the stresses or displacements on the loading factors.

Keywords**:** *Numerical Method, Solution, Differential Integration Equations, solution techniques, simulations methods.*

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1. INTRODUCTION

We have developed an analytical method for direct integration of the differential equilibrium and compatibility equations in terms of stresses, which does not make any use of auxiliary functions. The method was proposed by Prof. Vihak (Vigak) [2−6]. It is based on integration of the model equations, determination of relations between the stress components, and selection of the so-called key stresses. Consequently, the governing integro-differenttial equations are derived for the key stresses. To solve them, we propose a method for separation of variables. After determination of stresses, the displacements are found by integration of the Cauchy relations. Applying the method, we have already solved some problems; see Vihak et al. [1–11] for the details. This paper presents the solutions for a plane elasticity problem in a rectangle and the three-dimensional thermo elasticity problems in a half-space and an infinite layer.

2. SOLUTION FOR A PLANE PROBLEM OF ELASTICITY IN A RECTANGLE

Consider a plane elasticity problem in a rectangle $D = \{x \in [-a,a],\ y \in [-b,b]\}$. In the absence of body forces, the problem is governed by [12] the equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \qquad (1)$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

and the compatibility equation

$$\Delta\left(\sigma_x + \sigma_y\right) = 0\,, \qquad (2)$$

under the imposed tractions on the boundary:

$$\begin{aligned}
&\sigma_x\big|_{x=a} = -p_1(y), \quad \sigma_x\big|_{x=-a} = -p_2(y),\\
&\sigma_{xy}\big|_{x=a} = q_1(y), \quad \sigma_{xy}\big|_{x=-a} = q_2(y),\\
&\sigma_y\big|_{y=b} = -p_3(x), \quad \sigma_y\big|_{y=-b} = -p_4(x),\\
&\sigma_{xy}\big|_{y=b} = q_3(x), \quad \sigma_{xy}\big|_{y=-b} = q_4(x).
\end{aligned} \qquad (3)$$

Here $\sigma_x, \sigma_y, \sigma_{xy}$ are the stresses, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is

Laplace differential operator in dimensionless Cartesian coordinates system x, y. The tractions (3) should satisfy the general static conditions in integral form

$$\begin{aligned}
&\int_{-b}^{b}(p_2 - p_1)dy + \int_{-a}^{a}(q_3 - q_4)dx = 0,\\
&\int_{-a}^{a}(p_4 - p_3)dx + \int_{-b}^{b}(q_1 - q_2)dy = 0,\\
&\int_{-b}^{b}(p_2 - p_1)y\,dy + b\int_{-a}^{a}(q_3 + q_4)dx =\\
&= \int_{-a}^{a}(p_4 - p_3)x\,dx + a\int_{-b}^{b}(q_1 + q_2)dy.
\end{aligned}$$

Our solution strategy for problem (1)–(3) is the following:

• Select one key stress out of the three stress components and derive a governing equation for it.

• Equivalently replace the boundary conditions for separate stresses (3) by the conditions for a key stress only.

• Based on the form of the governing equation, construct a complete set of functions and use them to represent the key stress by convergent series.

• Calculate other stress tensor components.

We demonstrate our approach for the selected stress σy as a key one. Besides boundary conditions (3) for σxy at sides y = ±b , the key stress σy should satisfy the following boundary conditions:

$$\left.\frac{\partial \sigma_y}{\partial y}\right|_{y=b} = -\frac{dq_3}{dx}, \quad \left.\frac{\partial \sigma_y}{\partial y}\right|_{y=-b} = -\frac{dq_4}{dx} \qquad (4)$$

which follow from validness of Eqs (1) at y = ±b and the last Eq (3). Integration of the equilibrium equations yields the expressions for other two stresses in terms of the key stress; see Vihak et al. [11]:

$$\begin{aligned} 2\sigma_{xy} &= q_1 + q_2 - \int_{-a}^{a} \frac{\partial \sigma_y}{\partial y} \operatorname{sign}(x-\eta)\, d\eta, \\ 2\sigma_x &= -p_1 - p_2 + \frac{\partial}{\partial y}\big((a-x)q_1 - (a+x)q_2\big) + \\ &\quad + \int_{-a}^{a} \frac{\partial^2 \sigma_y}{\partial y^2} |x-\eta|\, d\eta. \end{aligned} \qquad (5)$$

Here,

$$\operatorname{sign}(t) = \begin{cases} 1, & t > 0, \\ -1, & t < 0. \end{cases}$$

In addition, the following integral conditions are valid:

$$\begin{aligned} 2\int_{-a}^{a} \sigma_y dx &= -\int_{-a}^{a} (p_3 + p_4)\, dx + \\ &\quad + \int_{-b}^{b} (q_2 - q_1) \operatorname{sign}(y-\xi)\, d\xi, \\ 2\int_{-b}^{b} y\sigma_x dy &= -\int_{-b}^{b} (p_1 + p_2) y dy + \\ &\quad + \int_{-a}^{a} (p_3 - p_4) |x-\eta| d\eta + \\ &\quad + \int_{-b}^{b} \big((x-a)q_1 + (x+a)q_2\big) dy - \\ &\quad - b\int_{-a}^{a} (q_3 + q_4) \operatorname{sign}(x-\eta) d\eta \end{aligned} \qquad (6)$$

The governing equation for σy follows from Eqs (1), (2), and second Eq (5):

$$2\frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial x^2}+\frac{1}{2}\int_{-a}^{a}\frac{\partial^4\sigma_y}{\partial y^4}|x-\eta|\,d\eta=$$
$$=\frac{1}{2}\frac{\partial^2}{\partial y^2}\left\{p_1+p_2+\frac{\partial}{\partial y}\left[(x-a)q_1+(x+a)q_2\right]\right\} \qquad (7)$$

We have proved that solving the original elasticity problem (1)–(3) is equivalent to solving Eq (7), under conditions (4)–(6). After separation of variables in Eq (7), we can find the key stress in the form of a decomposition by the complete orthogonal set of eigen- and associated functions

$$\left\{1,\ x, \cos\gamma_n\frac{x}{a}, \sin\lambda_n\frac{x}{a}\right\}:$$
$$\sigma_y=Y_0^1(y)+xY_0^2(y)+$$
$$+\sum_{n=1}^{\infty}\left[Y_n^1(y)\cos\gamma_n\frac{x}{a}+Y_n^2(y)\sin\lambda_n\frac{x}{a}\right], \qquad (8)$$

where

$$Y_0^1=\frac{1}{2a}\int_{-a}^{a}\sigma_y dx,$$
$$Y_0^2=\frac{3}{4a^3}\int_{-a}^{a}x\sigma_y dx,$$
$$Y_n^1=\frac{1}{a}\int_{-a}^{a}\sigma_y\cos\gamma_n\frac{x}{a}\,dx,$$
$$Y_n^2=\frac{1}{a\sin^2\lambda_n}\int_{-a}^{a}\sigma_y\sin\lambda_n\frac{x}{a}\,dx,$$
$$\gamma_n=\pi n,\ \tan\lambda_n=\lambda_n,\ \lambda_n>0,\ n=1,2,\ldots\ .$$

The associated functions {1, y } and {1, x } indicate the elementary solutions

$$\sigma_y^0=Y_0^1(y)+xY_0^2(y),$$
$$\sigma_x^0=X_0^1(x)+yX_0^2(x) \qquad (9)$$

in Eqs (8) and (5), which correspond to the resultant vector and moment, i.e., they depend on their non-self-equilibrated constituents. We have found these functions in a closed form [1] and performed their detailed analysis [7]. The self-equilibrated constituents of the stresses, which are specified by the eigen-functions, depend on the self-equilibrated tractions, satisfying the homogeneous integral equilibrium conditions thereby. We have computed these functions by means of an iterative algorithm; see Vihak et al. [3] for the details.

3. SOLUTION FOR A 3D PROBLEM OF THERMO ELASTICITY IN A HALF-SPACE AND AN INFINITE LAYER

Vihak demonstrated [10] for three-dimensional problems of mechanics that there are only three compatibility equations in terms of strains–not six, as it was believed before. They naturally coincide with three separate Saint-Venant's compatibility equations, depending on which three of the six Cauchy relations have been selected as the governing ones for calculating the displacements. On the basis of the three compatibility equations in terms of strains, the corresponding equations in terms of stresses can be written down. On a par with the equilibrium equations, they constitute a complete set of equations in terms of stresses for three-dimensional elasticity and thermo elasticity problems.

The three-dimensional thermo-stressed state in a half-space $D=\{x,y\in(-\infty,\infty),z\in[0,\infty)\}$ or a layer $D=\{x,y\in(-\infty,\infty),z\in[-1,1]\}$ is governed by the equilibrium equations

$$
\begin{aligned}
&\frac{\partial\sigma_x}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}=0,\\
&\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\sigma_{yz}}{\partial z}=0,\\
&\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}=0
\end{aligned}
\qquad (10)
$$

Cauchy relations

$$
\begin{aligned}
&e_x=\frac{\partial u_x}{\partial x},\quad e_y=\frac{\partial u_y}{\partial y},\quad e_z=\frac{\partial u_z}{\partial z},\\
&e_{xy}=\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x},\\
&e_{xz}=\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x},\\
&e_{yz}=\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}
\end{aligned}
\qquad (11)
$$

and physical relations for a linear thermoelastic solid

$$
\begin{aligned}
&Ee_x=\sigma_x-\nu(\sigma_y+\sigma_z)+\alpha ET,\\
&Ee_y=\sigma_y-\nu(\sigma_x+\sigma_z)+\alpha ET,\\
&Ee_z=\sigma_z-\nu(\sigma_x+\sigma_y)+\alpha ET,\\
&Ge_{xy}=\sigma_{xy},\quad Ge_{yz}=\sigma_{yz},\quad Ge_{xz}=\sigma_{xz},
\end{aligned}
\qquad (12)
$$

where, $\sigma_i,\sigma_{ij},\ e_i,e_{ij}\ (i=\{x,y,z\},\ i\neq j)$ are the stresses and strains, u_x, u_y, u_z are the displacements, T is a prescribed temperature field, E and ν denote the Young's modulus and Poisson's ratio, G and α denote the shear modulus and coefficient of thermal expansion. We have 15 Eqs (10)–(12) for 15 unknowns – 6 stresses, 6 strains, and 3 displacements, – complemented by the boundary conditions

$$\sigma_z\big|_{z=0} = -q(x,y),$$
$$\sigma_{xz}\big|_{z=0} = q_1(x,y), \qquad (13)$$
$$\sigma_{yz}\big|_{z=0} = q_2(x,y),$$

$$\sigma_z\big|_{z=\pm 1} = \begin{cases} -p_1(x,y), \\ -p_2(x,y), \end{cases}$$
$$\sigma_{xz}\big|_{z=\pm 1} = \begin{cases} q_1(x,y), \\ q_2(x,y), \end{cases} \qquad (14)$$
$$\sigma_{yz}\big|_{z=\pm 1} = \begin{cases} r_1(x,y), \\ r_2(x,y) \end{cases}$$

for a half-space and a layer, respectively. All the field variables should vanish at $\sqrt{x^2+y^2+z^2} \to \infty$.

To solve the problems (10)–(13); (10)–(12), (14) in terms of stresses, one should eliminate the strains and displacements. We calculate the displacements by the first three Eqs (11), to result in the first three Beltrami equations

$$\Delta\sigma_x = \frac{1}{1+\nu}\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(\sigma+\alpha ET),$$
$$\Delta\sigma_y = \frac{1}{1+\nu}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)(\sigma+\alpha ET), \qquad (15)$$
$$\Delta\sigma_z = \frac{1}{1+\nu}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma+\alpha ET),$$

where $\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$, $\sigma = \sigma_x+\sigma_y+\sigma_z$.

Therefore, we arrive at a closed set of six Eqs (10) and (15) for six stresses. Summing up equations (15) yields

$$\Delta\sigma = -\frac{2\alpha E}{1-\nu}\Delta T. \qquad (16)$$

Using the third Eq (10), we equivalently replace the boundary conditions (13)–(14) for shear stresses by those for derivatives of the normal stress:

$$\left.\frac{\partial\sigma_z}{\partial z}\right|_{z=0} = -\frac{\partial q_1}{\partial x}-\frac{\partial q_2}{\partial y}, \qquad (17)$$

$$\left.\frac{\partial\sigma_z}{\partial z}\right|_{z=1} = -\frac{\partial q_1}{\partial x}-\frac{\partial r_1}{\partial y},$$
$$\left.\frac{\partial\sigma_z}{\partial z}\right|_{z=-1} = -\frac{\partial q_2}{\partial x}-\frac{\partial r_2}{\partial y} \qquad (18)$$

We have solved the formulated problems (15), (16), (13), (17) for a half-space and (15), (16), (14), (18) for a layer, by applying a two-dimensional Fourier transform by x and y . The details are documented by Vihak et al. in [8, 9].

4. CONCLUSIONS

In this paper, by applying the proposed method of direct integration, we have derived an analytical solution of the plane elasticity problem in a rectangle. The solution is the sum of the self-equilibrated and elementary constituents. The elementary parts correspond to tension and bending. The self-equilibrated parts have an essential influence only close to the boundary, tending to zero when moving away from it. Also, we have solved the three-dimensional thermo elasticity problems in a half-space and a layer. The solutions in terms of stresses are constructed by direct consecutive integration of the governing second-order differential equations for the key stresses, without traditional use of intermediate functions.

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