A STUDY OF POLYNOMIALS AND LOCATION OF ZEROES IN OPEN AND CLOSED DISCS

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ABSTRACT

In this paper we are studding Study of polynomials and Location of zeroes in open and closed discs. Several analytical methods are used to derive estimates for part or all zeros of polynomials with complex coefficients and linear combinations of polynomials. Some results from Biernacki, Montel and Specht are strengthened or generalized. Some results are also obtained about the location of zeros of linear combinations of polynomials. The aim of this paper is to present a survey of certain results concerning the bounds for the modulii of the zeros of a polynomial. The problems of obtaining precise new limits, corrections and generalizations of some old results for the location of zeros of a polynomial are still of considerable interest. This topic is an active area of research in view of this fact and many yet unresolved questions. In the present paper, the zeros of the polynomials under consideration lie in annular regions. These areas can be improved, with additional conditions on the coefficient of repetition relation. For example, positivity allows correction of the annular region to some moon-like region.

Keyword : - *Closed Discs, Polynomial*, *Location,* *Expression etc.*

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1. INTRODUCTION

In the early 1900s, a Japanese mathematician Gustaf Hazelmer Anström, a Japanese mathematician and a mathematician, Sochi Kakaya, worked on a result that would give the location of a polynomial zero with non-monotonically increasing coefficients. Enestrestom was best known for creating the Enestr Indexom Index, which is used to identify Euler's writings [13]. Kakaya is most famous for solving the transportation problem, a very famous and important problem sought during the Second World War. Although both Enstrom and Kakeya proved the same result, they did so independently and both are credited for the proof [13], [28]. The Enestr theomKakeya theorem concerns the location of the zeros of a polynomial with increasing real, non-coefficient multiples.

The study of polynomial zeros has a rich history. In addition to many applications, the study has been the inspiration for much theoretical research. Algebraic and analytical methods for finding polynomial zeros, in general, can be complex. Traditionally, polynomials were objects of algebra. They were considered as algebraic expressions in an unknown and their zeros were seen as the roots of an equation. By the early part of the twentieth century, results on zero and critical points were included in books on algebra, despite the fact that they were of an analytical nature. Independently, in the late nineteenth century, polyandry attracted interest as a special type of work with excellent analytical properties. Historically speaking of the dates of that subject from the time when geometric representations of complex numbers were introduced into mathematics and the first contributors to the subject were Gauss and Cauchy. The subject has since been studied by many researchers. Due to applications in many fields like signal processing, communication theory, control theory, combinatorics, etc. there is always a need for better and better results in the subject.

The most complex problems of business and industry were called for the solution of equations and the introduction of literal symbols, thus algebra, which at the time pointed to the science of equations. Even in antiquity, solutions were for first order equations and for quadratic equations, which are the stumbling blocks of today's schoolchildren.

We recall here that an expression of the form

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 ,$$

where $a_n, a_{n-1}, \cdots, a_0$ are real or complex numbers with $a_n \neq 0$, is called a polynomial of degree n. If there is a value of $z = \alpha$ say, such that $P(\alpha) = 0$, then α is called the zero of polynomial $P(z)$. Enormous efforts were put into solving polynomial equations of degree higher than the second and only in sixteenth century were such solutions forthcoming for equations of the third and fourth degrees. Another three centauries were spent in vain efforts to get the solutions of polynomial equations of degree higher than the fourth. It required the generous of Abel and Galois to resolve this problem in it entirely. At the beginning of the nineteenth century, a young Norwegian mathematician, Neil Henry Abel mediated long and painstakingly on the problem and finally came to the conviction that equations of degree higher than fourth cannot, generally speaking, be solved by radicals. At about this time, another young mathematician Everest Galois of France took a new approach and proved a similar result.

1. **Fundamental Theorem of Algebra**
   Every polynomial
   $$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 ,$$
   where $a_n, a_{n-1}, \cdots, a_0$ are complex numbers with $a_n \neq 0$, of degree $n \geq 1$ has n zeros.
2. **Rouches Theorem**
   If $f(z)$ and $g(z)$ are analytic inside and on a closed contour C and $|g(z)| < |f(z)|$ on C, then $f(z)$ and $f(z) \pm g(z)$ has the same number of zeros inside C.
3. **Gauss-Lucas Theorem**
   Any circle C which encloses all the zeros of a polynomial $P(z)$ also encloses all the zeros of its derivative $P'(z)$.
4. **Maximum Modulus Principle**
   If $f(z)$ is analytic and non-constant in region D, then its absolute value $|f(z)|$ is maximum on D but not inside D.
5. **Schwarz Lemma**
   If $f(z)$ is analytic function, regular for $|z| \leq R$ and $|f(z)| \leq M \; for \; |z| = R$ and $f(0) = 0$, then
   $$\left|f(re^{i\theta})\right| \leq \frac{Mr}{R} , \qquad 0 \leq r \leq R \;\text{ and }\; 0 \leq \theta \leq 2\pi.$$

Proof of Theorem 1.3 Let f have n zeros in the disk $|z| \leq \delta R$, say $a_1, a_2, \ldots, a_n$. Then for $1 \leq k \leq n$ we have $|a_k| \leq \delta R$, or $\frac{R}{|a_k|} \geq \frac{1}{\delta}$. So

$$\sum_{k=1}^{n} \log \frac{R}{|a_k|} = \log \frac{R}{|a_1|} + \log \frac{R}{|a_2|} + \cdots + \log \frac{R}{|a_n|} \geq n \log \frac{1}{\delta}. \tag{1}$$

By Jensen's Formula, we have

$$\begin{aligned}
\sum_{k=1}^{n} \log \frac{R}{|a_k|} &= \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| \, d\theta - \log |f(0)| \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \log M \, d\theta - \log |f(0)| \\
&= \log M - \log |f(0)| \\
&= \log \frac{M}{|f(0)|}. \qquad (2)
\end{aligned}$$

Combining (1) and (2) gives

$$n \log \frac{1}{\delta} \leq \sum_{k=1}^{n} \log \frac{R}{|a_k|} \leq \log \frac{M}{|f(0)|},$$

or

$$n \leq \frac{1}{\log 1/\delta} \log M|f(0)|.$$

Since n is the number of zeros of f in $|z| \leq \delta R$, the result follows.

2. THEOREM

THEOREM 3.1: 1 If all the zeros of the polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 ,$$

of degree n, lie in the circle $|z| \leq R$, then all the zeros of its derivative $P'(z)$ also lie in

$$|z| \leq R.$$

PROOF OF THEOREM 3.1. Let $z_1, z_2, \cdots, z_n$ be the zeros of $P(z)$, then

$$|z_j| \leq R, \quad \text{for all} \quad j = 0,1, \cdots, n$$

and

$$P(z) = a_n (z - z_1)(z - z_2) \ldots (z - z_n)$$
$$= a_n \prod_{j=1}^{n} (z - z_j).$$

This gives,

$$\mathrm{Log} P(z) = \mathrm{Log} a_n + \sum_{j=1}^{n} \mathrm{Log}(z - z_j).$$

Differentiating both sides with respect to z, we get

$$\frac{P'(z)}{P(z)} = \sum_{j=1}^{n} \frac{1}{(z - z_j)}. \tag{3.1}$$

Now let w be a zero of $P'(z)$. If w is also a zero of $P(z)$, then $|w| \leq R$ and therefore the result follows. So we suppose that w is not a zero of $P(z)$, then $P(w) \neq 0$, so that $w \neq z_j$, for any $j = 1,2, \cdots, n$, and $P'(w) = 0$. Now from (3.1) we have

$$\sum_{j=1}^{n} \frac{1}{(w - z_j)} = \frac{P'(w)}{P(w)} = 0.$$

This gives

$$\sum_{j=1}^{n} \frac{1}{\overline{w - z_j}} = 0,$$

which implies

$$\sum_{j=1}^{n} \frac{(w - z_j)}{(\overline{w} - \overline{z}_j)(w - z_j)} = 0,$$

or equivalently

$$\sum_{j=1}^{n} \frac{(w - z_j)}{|w - z_j|^2} = 0,$$

so that

$$\sum_{j=1}^{n} \frac{w}{|w - z_j|^2} = \sum_{j=1}^{n} \frac{z_j}{|w - z_j|^2},$$

which gives

$$\left| w \sum_{j=1}^{n} \frac{1}{|w - z_j|^2} \right| = \left| \sum_{j=1}^{n} \frac{z_j}{|w - z_j|^2} \right|$$

and hence

$$|w| \sum_{j=1}^{n} \frac{1}{|w - z_j|^2} \le \sum_{j=1}^{n} \frac{|z_j|}{|w - z_j|^2}$$
$$\le R \sum_{j=1}^{n} \frac{1}{|w - z_j|^2},$$

from which we conclude that $|w| \le R$. Since w is an arbitrary zero of $P'(z)$, it follows that all the zeros of $P'(z)$ lie in the circle $|w| \le R$ and this completes the proof of Theorem 3.1.

COROLLORY 3.1. If all the zeros of the polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

of degree n lie in the circle $|z - c| \le R$, then all the zeros of $P'(z)$ also lie in the circle

$$|z - c| \le R.$$

PROOF OF COROLLORY 3.1. Since the polynomial $P(z)$ has all its zeros in $|z - c| \le R$, it follows that all the zeros of the polynomial $P(z + c)$ lie in $|z| \le R$. Hence by Theorem 3.1, all the zeros of $P'(z + c)$ also lie in $|z| \le R$. Replacing z by $z - c$, we conclude that all the zeros $P'(z)$ lie in $|z - c| \le R$, which is the desired result.

3. FINDING THE NUMBER OF ZEROS OF A POLYNOMIAL IN THE CLOSED DISK

Find the number of zeros of $f(z)=z^6-5z^4+3z^2-1f(z)=z^6-5z^4+3z^2-1$ in $|z|\le1|z|\le1$.
My attempts have not gotten far.
I know we can examine the related equation $f(w)=w^3-5w^2+3w-1f(w)=w^3-5w^2+3w-1$ in $|w|\le1|w|\le1$, letting $w=z^2w=z^2$.
It is clear that $f(w)=0f(w)=0$ for $|w|=1|w|=1$ if and only if $w=-1w=-1$.
My main problem is that this seems obviously like an application of Rouché's theorem, where we let $g(w)=5w2g(w)=5w2$, whereupon we have the relation $|f(w)|<|g(w)||f(w)|<|g(w)|$ for all $|w|=1|w|=1$, with the sole exception that equality holds when $w=-1w=-1$.
I would like to use Rouché to equate the number of zeros of ff and gg, but my understanding of Rouché is that the inequality should be strict with no exceptions.
Rouche's theorem.
if $|g(z)|>|h(z)||g(z)|>|h(z)|$ for all zz along some closed contour. $g(z)+h(z)g(z)+h(z)$ has as many zeros inside the contour as $g(z)g(z)$ has inside the contour.
But what about when $|g(z)|=|h(z)||g(z)|=|h(z)|$? Then there is the possibility that some zeros lie on the contour.
In this case let $g(z)=5z^4,h(z)=z^6+3z^2-1g(z)=5z^4,h(z)=z^6+3z^2-1$ on the contour $|z|=1|z|=1$
$|g(z)|=5,|h(z)|\le5|g(z)|=5,|h(z)|\le5$

There will be 44 zeros in the disk |z|≤1|z|≤1 with some possibly on the contour.

4. SUMMARY

In solving an nth order linear differential equation with constant coefficients, it is often an issue whether its solutions remain bounded as the variable tends to infinity. This is so, provided that all the zeros of its characteristic polynomial have negative real parts. A polynomial with this property is said to be stable. The Routh-Hurwitz Problem is to determine what conditions are equivalent to stability. For real linear, quadratic and cubic polynomials, the criterion is reasonably straightforward. All that is required for a quadratic is that its coefficients have the same sign, all positive or all negative. For the cubic, $z^3 + bz^2 + cz + d$, it is necessary and sufficient that all coefficients are positive and also that bc > d. We observe that it suffices to consider polynomials with real coefficients.

In the case of real polynomials, we can apply the intermediate value theorem which states that any interval (a, b) of the polynomial p (x) has at least one zero for which p (a) and p (b) Nonzero numbers are opposite signs. There are other tests of increasing sophistication. One of the simplest and convenient is Descartes's Rules of Signs which asserts that for a real polynomial, the number of positive zeros of (zero) across its non-axial coefficient changes when reading from left to right. Can not And the number of negative zeros cannot exceed the number of change of signs in P (-x).

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