A STUDY ON THE LINEAR DIFFERENCE EQUATIONS FOR (m, n-m)- BOUNDARY VALUE PROBLEMS USING GREEN'S FUNCTIONS

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ABSTRACT

The aim of this paper is to introduce that the concept of linear difference equation of (m,n-m) boundary value problem using the Green's function. Also we characterized the pair of conjugate points and the eigen value for linear difference using Green's function.

Keywords –*Green's function, boundary value, linear difference equation, eigen value, conjugate points*

INTRODUCTION:

Difference equations appear naturally as discrete analogues and numerical solutions of differential equations and delay differential equations having applications in biology, ecology, physics, etc. Difference equations have also become very important due to their applications to numerically solve ODE's and PDE's.

Recently there has been interest in difference equations, partly due to Hartman's paper[5]. We use much of his notation in paper. Here we will be concerned with the n^{th} order linear difference equation

$$Qu(p) = \sum_{i=0}^{n} \alpha_i(p)u(p+i) = 0$$

where p is in the interval $I = [a,b] = \{a, a+1, \dots b\}$ where a & b are integers. We assume that $\alpha_0(p) = 1$ for $p \in I$ and that

$$(-1)^n \alpha_0(p) > 0, \qquad p \in I$$

Also, the Green's function for the boundary value problem

$$Y^{2n}(x) = 0$$
$$Y(a_1) = Y(a_2) = \cdots = Y(a_{2n}) = 0$$

has been explicitly obtained in Das and Vatsala [4]. We obtain a necessary condition for a solution of the differential equation

$$Y^{2n} + (-1)^{n-1}PY = 0$$

to satisfy the boundary conditions with a, b replaced by α, β respectively, where $a \leq \alpha < \beta \leq b$.

From this, we conclude that $\Phi_b(a)$ is a monotone function of a if $p < 0$ and therefore a non-trivial solution $y(x)$ of necessary condition can have atmost one n^{th} order zero. Hence we assume that p is positive somewhere in $[a, b]$.

Definition: 1.1

A difference equation is also called as a recurrence equation, which is an equation that defines a sequence recursively; each term of the sequence is defined as a function of the previous terms of the sequence:

$$X_n = f(X_{n-1}, X_{n-2}, \ldots X_0)$$

Definition: 1.2

Linear differential equations are differential equations having solutions which can be added together in particular linear combinations to form further solutions. They equate 0 to a polynomial that is linear in the value and various derivatives of a variable; its linearity means that each term in the polynomial has degree either 0 or 1.

Linear differential equations can be ordinary (ODEs) or partial (PDEs). The solutions to linear differential equations form a vector space.

Definition: 1.3

A linear differential operator $L = L(x)$ acting on the collection of distributions over a subset Ω of some euclidean space $\mathbb{R}^n$, a Green's function $G = G(x,s)$ at the point $s\epsilon\Omega$ corresponding to L is any solution of

$$L\, G(x,s) = \delta(x-s)$$

whereδ denotes the delta function.

Definition: 1.4

A boundary value problem is a problem, typically an ordinary differential equation or a partial differential equation, which has values assigned on the physical boundary of the domain in which the problem is specified. For example,

$$\begin{cases} \dfrac{\delta^2 u}{\delta t^2} - \nabla^2 u = f & in\ \Omega \\ u(0,t) = u_1 & on\ \delta\Omega \\ \dfrac{\delta u}{\delta t}(0,t) = u_2 & on\ \delta\Omega \end{cases}$$

where$\delta\Omega$ denotes the boundary of Ω, is a boundary problem.

Definition: 1.5

An eigen vector or characteristic vector of a linear transformation is a non-zero vector that does not change its direction when that linear transformation is applied to it. If T is a linear transformation from a vector space V over a field F into itself and v is a vector in V that is not the zero vector, then v is an eigen vector of T if $T(v)$ is a scalar multiple of v. This condition can be written as the equation $AX = \lambda X$ where λ is a scalar in the field F, known as the eigen value, characteristic value, or characteristic root associated with the eigen vector v.

Definition: 1.6

A function is said to be monotone function, if it is either increasing or decreasing. A function $Y = g(x)$ is called monotonic on the interval I if it is either decreasing, increasing or constant.

(1)A function $g(x)$ is called decreasing if $x_1 > x_2$ and $g(x_1) < g(x_2)$ for all x_1, x_2in I.

(2) A function $g(x)$ is called increasing if $x_1 > x_2$ and $g(x_1) > g(x_2)$ for allx_1, x_2in I.

(3)A function $g(x)$ is called constant if $x_1 > x_2$ and $g(x_1) = g(x_2)$ for all x_1, x_2in I.

Theorem : 1.7

Let m be a positive integer.Then

$$-\left(\frac{(r-x)(y-s)}{y-x}\right)^m \sum_{j=0}^{m-1} \binom{m-1+j}{j} (s-r)^{m-1-j} \left(\frac{(y-r)(s-x)}{y-x}\right)^j - (-1)^{m-1}(r-s)^{2m-1} = \left(\frac{(s-x)(y-r)}{y-x}\right)^m \sum_{j=0}^{m-1} \binom{m-1+j}{j} (r-s)^{m-1-j} \left(\frac{(y-s)(r-x)}{y-x}\right)^j \quad \text{-----------}(*)$$

Proof:

Equation $(*)$ is immediate for $m = 1$

Assume it is true for $m = p$

Then, in view of the identity

$$\left(\frac{(r-x)(y-s)}{y-x}\right) = (r-s) + \left\{\frac{(s-x)(y-r)}{y-x}\right\}$$

The LHS of $(*)$ for $m = p + 1$ can be written as

$\left(\frac{(r-x)(y-s)}{y-x}\right)^p\left\{\sum_{j=0}^{p}\binom{p+j}{j}(s-r)^{p-j}\left(\frac{(y-r)(s-x)}{y-x}\right)^{j+1}-\right.$
$\left.\sum_{j=0}^{p}\binom{p+j}{j}(s-r)^{p+1-j}\left(\frac{(y-r)(s-x)}{y-x}\right)^{j}\right\}-(-1)^p(r-s)^{2p+1}=$
$\left(\frac{(r-x)(y-s)}{y-x}\right)^p\left\{\binom{2p}{p}\left(\frac{(y-r)(s-x)}{y-x}\right)^{p+1}-\sum_{j=0}^{p}\binom{p-1+j}{j}(s-r)^{p+1-j}\left(\frac{(y-r)(s-x)}{y-x}\right)^{j}\right\}-$
$(-1)^p(r-s)^{2p+1}$
$=$
$\left(\frac{(r-x)(y-s)}{y-x}\right)^p\left\{\binom{2p}{p}\left(\frac{(y-r)(s-x)}{y-x}\right)^{p+1}-\binom{2p-1}{p}(s-r)\left(\frac{(y-r)(s-x)}{y-x}\right)^{p}\right\}-$
$\left(\frac{(s-x)(y-r)}{y-x}\right)^p\sum_{j=0}^{p-1}\binom{p-1+j}{j}(r-s)^{p+1-j}\left(\frac{(y-s)(r-x)}{y-x}\right)^{j}$

using the induction hypothesis.
Again, writing

$$\binom{p-1+j}{j}=\binom{p+j}{j}-\binom{p-1+j}{j-1}, \quad j=1,2,\ldots,p-1$$

And rearranging terms in the above we get

$$\left(\frac{(s-x)(y-r)}{y-x}\right)^{p+1}\left\{\left(\frac{(r-x)(y-s)}{y-x}\right)^p\binom{2p}{p}+\sum_{j=0}^{p-1}\binom{p+j}{j}(r-s)^{p-j}\left(\frac{(y-s)(r-x)}{y-x}\right)^p\right\}$$

which is,

$$\left(\frac{(s-x)(y-r)}{y-x}\right)^{p+1}\sum_{j=0}^{p}\binom{p+j}{j}(r-s)^{p-j}\left(\frac{(y-s)(r-x)}{y-x}\right)$$

Hence the proof

Theorem: 1.8

If there exists a pair of conjugate points on $[x,y]$ with respect to $Y^{(2n)}+(-1)^{n-1}PY=0$, then
$\int_x^y(s-x)^{2n-1}(y-s)^{n-1}p(s)ds\geq(2n-1)[(n-1)!]^2(y-x)^{2n-1}$--(1)

Proof:

Consider the auxiliary equation

$$Y^{(2n)}+(-1)^{n-1}PY=0\text{----(2)}$$

The differential equation (2) has a pair of conjugate points on $[x,y]$say x',y'.
Now Consider the eigen value problem,

$$Y^{(2n)}+(-1)^{n-1}\lambda PY=0\text{----(3)}$$
$$Y(x')=\cdots=Y^{(n-1)}(x')=Y(y')=\cdots=Y^{(n-1)}(y')=0$$

The problem (3) is equivalent to the integral equation,

$$u(r)=\lambda\int_x^y m(r,s)u(s)ds\text{----(4)}$$

where,

$$m(r,s)=(-1)^{n-1}g_{2n}(r,s)[p(r)p(s)]^{1/2}$$
$$u(r)=\left(p(r)\right)^{\frac{1}{2}}Y(r) \qquad \text{----(5)}$$

And $g_{2n}(r,s)$ is the Green's function for

$$Y^{(2n)}(r)=0$$

and$Y^{(m)}(x)=0=Y^{(m)}(y)$, $m=0,1,2,\ldots n-1$
with $x=x',y=y'$.

Since the m^{th} eigen value λ_m of eqn (3) is the minimum of $\int_{x'}^{y'}\left(w^{(n)}\right)^2dx/\int_{x'}^{y'}pw^2dx$,
where $w\in D^n[x',y']$, satisfies the boundary condition in eqn (3) as well as the Orthogonality conditions related to Courant's well known mini-max principle, $m(r,s)$ is non-negative definite.
This leads to,

$$1 \leq \sum_{m=1}^{\infty} \frac{1}{\lambda_m} = \int_{x'}^{y'} |g_{2n}(s,s)|P(s)ds$$

In view of thrm (1) and

$$[x', y'] \subseteq [x, y]$$

Therefore eqn (3) yields (1).

Hence the proof

Theorem: 1.9

Let $p(r) = (\alpha r + \beta)$ in $Y^{(2n)} + (-1)^{n-1}PY = 0$ be positive on $[x, y]$. If there exists a pair of conjugate points with respect to $Y^{(2n)} + (-1)^{n-1}PY = 0$on $[x, y]$, then

$$\left[\alpha\left(\frac{x+y}{2}\right) + \beta\right] > 2^n(n-1)!\left(\prod_{i=n-1}^{n-1}(2i+1)\right)/(y-x)^{2n}$$

Proof:

In this special case in view of the strict inequality in thrm (2) of eqn (1), we have

$$\int_x^y (y-s)^{2n-1}(s-x)^{2n-1}(\alpha s + \beta)ds > (2n-1)((n-1)!)^2(y-x)^{2n-1}$$

The value of the integral is

$$(y-x)^{4n-1}\{[(2n-1)!]^2/(4n-1)!\}\left[\frac{\alpha(x+y)}{2} + \beta\right]$$

Suppose if $p(x) > 0, p(y) < 0$ then

$$\alpha\left(\frac{\beta}{\alpha} + x\right)^{2n+1}\left\{\sum_{j=0}^{2n-1} \frac{j!}{(2n+j+1)!}(y-x)^{2n-1-j}\left(\frac{\beta}{\alpha} + x\right)^j\right\} \geq \frac{(n-1)!\,(y-x)^{2n-1}}{\prod_{i=2}^{n}(2n-i)}$$

Hence the proof

Theorem: 1.10

Let $[x, y]$ have a pair of conjugate points with respect to the differential equation $Y^{(2n)} + (-1)^{n-1}PY = 0$, where $p(r)$ is continuous, positive and convex. Then

$[p(x) + p(y)] > 2^{n+1}(n-1)!\,(\prod_{i=n-1}^{2n-1}(2i+1))/(y-x)^{2n}$----(1)

If however, $p(x) > 0$and $p(y) < 0$, then

$\frac{[p(x)]^{2n+1}}{[p(y)-p(x)]^{2n}}\sum_{j=0}^{2n-1}\frac{j!}{(2n+j+1)!}\left\{\frac{p(x)}{p(y)-p(x)}\right\}^j \geq \frac{(n-1)!}{\prod_{i=2}^{n}(2n-i)(b-a)^{2n+1}}$----(2)

Proof:

Consider the auxiliary equation,

$Y^{(2n)} + (-1)^{n-1}q(r)Y = 0$----(3)

where$q(r) = \alpha r + \beta$ with

$$\alpha = [p(y) - p(x)]/(y-x)$$

and$\beta = [p(x)y - p(y)x]/(y-x)$

Then by a known theorem, we will have a pair of conjugate points with respect to eqn (3).

Hence eqn (1) & (2) follow from eqn (1) & (2) of thrm (2) for appropriate values of α & β.

Hence the proof

Theorem: 1.11

Let $p(r) > 0$ in $Y^{(2n)} + (-1)^{n-1}PY = 0$ be concave $[x, y]$. If $[x, y]$ has a pair of conjugate points with respect to $Y^{(2n)} + (-1)^{n-1}PY = 0$, then

$$p\left[\frac{x+y}{2}\right] > 2^n(n-1)!\left(\prod_{i=n-1}^{2n-1}(2i+1)\right)/(y-x)^{2n}$$

Proof:

Consider the auxiliary equation

$$Y^{(2n)} + (-1)^{n-1}q(r)Y = 0$$

withα, β such that the graph of $q(r)$ is a supporting line for the graoh of $p(r)$.

As in thrm (4), we get

$$\{q(x)+q(y)\} > 2^{n+1}(n-1)!\left(\prod_{i=n-1}^{2n-1}(2i+1)\right)/(y-x)^{2n}$$

Since this inequality is true for any supporting line, it is also true for that supporting line which is the graph of $q(r)$ for which $\{q(x)+q(y)\}$ is a minimum.

If Q stands for the set of all linear functions $q(r)$ as above, then

$$q(r) = (r-r_1)p'(r_1)+p(r_1) \text{for some } r_1 \in [x,y]$$

Thus,

$$\min_{q\in Q}\{q(x)+q(y)\} = \min_{r_1\in[x,y]}[(x+y-2r_1)p^{'}(r_1)+2p(r_1)]$$
$$= 2p[\tfrac{x+y}{2}]$$

Hence the proof

Theorem: 1.12

[variation of constants formulas]

(a) The solution of

$$Qu(p) = h(p)$$
$$u(x+i) = 0, \qquad 0 \le i \le n-1$$

is given by

$$u(p) = \sum_{s=x}^{p-1} U(p,s)h(s)\,, \qquad x \le p \le y+n$$

where$u(x) = 0$ is understood and $U(p,s)$ for each fixed s is the solution of $Qu(p) = 0$ satisfying

$$U(s+i,s) = 0, \qquad 1 \le i \le n-1$$
$$U(s+n,s) = 1.$$

Proof:

By Hartman's Paper, it is noted there that $h(p)$ can be defined arbitralily outside of $[x,y]$.

Assume $x+n \le q \le y+n$, then

$$u(q) = \sum_{s=x}^{q-1} U(q,s)h(s)$$

$$u(q-1) = \sum_{s=x}^{q-1} U(q-1,s)h(s) + U(q-1,q)h(q)$$

.
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$$u(q-n) = \sum_{s=x}^{q-1} U(q-n,s)h(s) + U(q-n,q)h(q) + \cdots + U(q-n,q-n+1)h(q)$$

Using the boundary conditions of $U(p,s)$, multiply each sides of the above equations by $\alpha_n(q-n), \alpha_{n-1}(q-n), \dots, \alpha_0(q-n)$, respectively.

$$\alpha_n(q-n)u(q) = \alpha_n(q-n)\sum_{s=x}^{q-1} U(q,s)h(s)$$

$$\alpha_{n-1}(q-n)u(q-1) = \alpha_{n-1}(q-n)\sum_{s=x}^{q-1} U(q-1,s)h(s) + U(q-1,q)h(q)$$

.
.
.

$$\alpha_0(q-n)u(q-n) = \alpha_0(q-n)\sum_{s=x}^{q-1} U(q-n,s)h(s) + U(q-n,q)h(q) + \cdots + U(q-n,q-n+1)h(q)$$

Adding all the above equations and finally replacing $q-n$ by $p$we get that

$$Qu(p) = h(p), \qquad x \le p \le y$$

Thus, it is easy to check that

$$u(p) = \sum_{s=x}^{p-1} U(p,s)h(s)$$

which satisfies the correct initial conditions.

(b)The solution of

$$Qu(p) = h(p)$$
$$u(y+i) = 0, \qquad 1 \le i \le n$$

is given by

$$u(p) = \sum_{s=p+1}^{y+n} V(p,s)\frac{h(s-n)}{\alpha_0(s-n)}, \qquad x \le p \le y+n$$

where$u(y+n) = 0$ is understood and $V(p,s)$ for each fixed s is the solution of $Qu(p) = 0$ satisfying

$$V(s-i,s) = 0, \qquad 1 \le i \le n-1$$
$$V(s-n,s) = 1.$$

Proof:

By Hartman's Paper, it is noted there that $h(p)$ can be defined arbitralily outside of $[x,y]$.

Assume $x+n \le q \le y+n$, then

$$u(q) = \sum_{s=q+1}^{y+n} V(q,s)\frac{h(s-n)}{\alpha_0(s-n)}$$

$$u(q-1) = \sum_{s=q+1}^{y+n} V(q-1,s)\frac{h(s-n)}{\alpha_0(s-n)} + V(q-1,q)\frac{h(q-n)}{\alpha_0(q-n)}$$

.
.
.

$$u(q-n) = \sum_{s=q+1}^{y+n} V(q-n,s)\frac{h(s-n)}{\alpha_0(s-n)} + V(q-n,q)\frac{h(p-n)}{\alpha_0(p-n)} + \cdots + V(q-n,q-n+1)\frac{h(q-2n)}{\alpha_0(q-2n)}$$

Using the boundary conditions of $U(p,s)$, multiply each sides of the above equations by $\alpha_n(q-n), \alpha_{n-1}(q-n), \dots, \alpha_0(q-n)$, respectively.

$$\alpha_n(q-n)u(q) = \alpha_n(q-n)\sum_{s=q+1}^{y+n} V(q,s)\frac{h(s-n)}{\alpha_0(s-n)}$$

$$\alpha_{n-1}(q-n)u(q-1) = \alpha_{n-1}(q-n)\sum_{s=q+1}^{y+n} V(q-1,s)\frac{h(s-n)}{\alpha_0(s-n)} + V(q-1,q)\frac{h(q-n)}{\alpha_0(q-n)}$$

.
.
.

$$\alpha_0(q-n)u(q-n) = \alpha_0(q-n)\sum_{s=q+1}^{y+n} V(q-n,s)\frac{h(s-n)}{\alpha_0(s-n)} + V(q-n,q)\frac{h(p-n)}{\alpha_0(p-n)} + \cdots + V(q-n,q-n+1)\frac{h(q-2n)}{\alpha_0(q-2n)}$$

Adding all the above equations and finally replacing $q-n$ by $p$we get that

$$Qu(p) = h(p), \qquad x \le p \le y$$

Thus, it is easy to check that

$$u(p) = \sum_{s=p+1}^{y+n} V(p,s)\frac{h(s-n)}{\alpha_0(s-n)}$$

which satisfies the correct initial conditions.

Hence the proof

CONCLUSION:

The Difference Equation of Hartman's paper [5] deals with the green's function of linear difference equation for (m, m-n) variables for which the green's function have been explicitly obtained in Das and Vatsala [4]. From this paper we have proved that $\Phi_b(a)$ is a monotone function of a if $p < 0$ and therefore a non-trivial solution $y(x)$ of necessary condition can have atmost one n^{th} order zero. Hence we assume that p is positive somewhere in $[a, b]$.

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