A Study of Mathematical Modelling Using Differential Equations

Sanjay Kumar Rai[1], Dr. Bhupendra[2]

[1]Research Scholar of Sri Satya Sai University of Technology & Medical Sciences, Sehore, M.P., India.
[2]Research Supervisor of Sri Satya Sai University of Technology & Medical Sciences, Sehore, M.P., India.

Abstract

In mathematics, a differential equation is an equation that relates one or more functions and their derivatives. In applications, the functions generally represent physical quantities, the derivatives represent their rates of change, and the differential equation defines a relationship between the two. Such relations are common; therefore, differential equations play a prominent role in many disciplines including engineering, physics, economics, and biology. Mainly the study of differential equations consists of the study of their solutions (the set of functions that satisfy each equation), and of the properties of their solutions. Only the simplest differential equations are solvable by explicit formulas; however, many properties of solutions of a given differential equation may be determined without computing them exactly. Often when a closed-form expression for the solutions is not available, solutions may be approximated numerically using computers. The theory of dynamical systems puts emphasis on qualitative analysis of systems described by differential equations, while many numerical methods have been developed to determine solutions. Differential equations can be divided into several types. Apart from describing the properties of the equation itself, these classes of differential equations can help inform the choice of approach to a solution. Commonly used distinctions include whether the equation is ordinary or partial, linear or non-linear, and homogeneous or heterogeneous. This list is far from exhaustive; there are many other properties and subclasses of differential equations which can be very useful in specific contexts.

Keywords: *Differential Equation, Expression, Ordinary Equation, Homogeneous or Heterogeneous Equation.*

1. INTRODUCTION

Principles of Physics such as Newton's second law of motion, principle of least action, Hamilton's principle and so on can be expressed in a beautiful manner using the mathematical language of differential equations. Differential equations also help Scientists, Engineers and Technologists to understand many practical problems such as problems of chemical kinetics, problems of ecology, and problems of finance and so on, because they serve as mathematical models built on meaningful mathematical principles. There are several aspects of studies of differential equations such as

(i) Mathematical modelling of practical problems using differential equations.

(ii) Existence and Uniqueness of solutions.

(iii) Stability and Controllability of the dynamical system.

(iv) Bifurcation and Chaos.

(v) Numerical solutions of differential equation, their Stability and Convergence.

(vi) Series solutions yielding useful approximate solution when subjected to Pade approximation and/or Asymptotic approximation and so on.

The aim of the present thesis is to work on the (vi) and the last objective mentioned above. For the purpose of getting a good motivation, let us describe a problem of mechanics based on Newton's second law of motion in the language of differential equation and describe three methods to compute exact solution, namely, Adomian decomposition series method, Laplace transform method and Laplace decomposition series method.

2. NAVIER-STOKES EQUATIONS FOR THE FLOW OF A VISCOUS INCOMPRESSIBLE FLUID

Let u (x, y, z, t), v (x, y, z, t), w (x, y, z, t) and p (x, y, z, t) denote respectively the three velocity components and pressure at the point (x, y, z) at time t in a fluid with constant density p and viscosity coefficient μ. Then the equation of continuity, which expresses the fact that the amount of fluid entering a unit volume per unit time is the same as the amount of the fluid leaving it per unit time, is given by.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

The equations of motion are obtained from Newton's second law of motion which states that the product of mass and acceleration of any fluid elements is equal to the resultant of all the external body forces acting on the element and to the surface forces acting on the fluid volume due to the action of the remaining fluid on the same element. The equations of motion, known as Navier-Stakes equations, for the flow of a Newtonian viscous incompressible fluid are

$$p\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=X-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right),$$

$$p\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=Y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right),$$

$$p\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=Y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right),$$

If the external body forces X, Y, Z form a conservative system, there exists a potential function Ω such that

$$X=-\frac{\partial \Omega}{\partial x},\ Y=-\frac{\partial \Omega}{\partial y},\ \ Z=-\frac{\partial \Omega}{\partial z},$$

$$X-\frac{\partial p}{\partial x}=-\frac{\partial}{\partial x}(\Omega+p),\ Y-\frac{\partial p}{\partial y}=-\frac{\partial}{\partial y}(\Omega+p),\ Z-\frac{\partial p}{\partial z}=-\frac{\partial}{\partial z}(\Omega+p)$$

so that p is effectively replaced by p +Ω

If X, Y, Z are known or are absent give a system of four coupled nonlinear partial differential equations for the four unknown function, u, v, w, and p. These equations have to be solved subject to certain initial conditions giving the motion of the fluid at time t = 0 and certain prescribed boundary conditions on the surfaces with which the fluid may be in contact or conditions which may hold at very large distances from the surfaces. Usually, the boundary conditions are provided by the no-slip condition according to which both tangential and normal components of the fluid velocity vanishes at all points of the surfaces of the stationary bodies with which the fluid may be in contact.

3. SOLUTIONS OF INTEGRO-DIFFERENTIAL EQUATIONS

Several researches have been studying different types of integral-differential equations which have applications such as heat transfer, neutron diffusion, ecology of coexistence of two or more species and so on. These kinds of equations with or without difference terms can also be found in many Science and Engineering applications serving as models dealing with advanced integral equations. Also, in the recent times, many researchers are interested in solving a singularly perturbed second order integro-differential-difference equation with one interval condition involving left extreme point of the boundary and another boundary condition at the right extreme of the boundary. The singular perturbation parameter and the delay parameter are selected as small as possible. Such problems play an important role in variety of physical problems such as micro scale heat transfer, diffusion in polymers, control of chaotic systems and so on (relevant references quoted). In the present study, we formulate a different problem, namely, an integro differential-difference equation with differential order one or two and difference of order one or two with initial interval condition. This can be done by considering the following integro-differential-difference equation of order:

$$\epsilon\, u''(t) = cu'(t) - [f(t) + F_1(u(t-\omega), u'(t-\omega)) + \int_0^t G(u(t_1 - \omega_k))dt_1],$$

where we allow either = 0 or c = 0 and ωk = ω or 2ω. Then in general, it will be a differential-difference equation of order (2, 2). Further, we work with the following initial interval condition:

$$u(t) = \lambda, \qquad t \in (0, 2\omega).$$

In continuation with the studies with a common objective to demonstrate the fact that Laplace decomposition method shows flexibility as well as provides convenience in the computation of analytical solutions for both linear and nonlinear problems, in the following types of integro-differential difference equations are considered for further study:

1. First order differential and first order difference integro-differential-difference equation of the following simple type:

$$u'(t) = k\, u'(t-\omega) + a\, f(t-\omega) + b\, g(u(t-\omega)) + c\int_0^t h(u(t_1-\omega))dt_1, \quad t > \omega,$$

$$u(t) = \lambda, \qquad 0 \le t \le \omega.$$

In the above equations, a, b, k, λ and $c \neq 0$ are known constants, f(t), g(u(t)) and h(u(t)) are given linear or nonlinear functions depending upon the particular problem discussed.

2. Second order differential and second order difference integro-differential-difference equation of the following simple type:

$$u''(t) = a\, f(t-\omega) + b\, g(u(t-\omega)) + c\int_0^t h(u(t_1-2\omega))dt_1, \quad t > 2\omega,$$

$$u(t) = \lambda, \qquad 0 \le t \le 2\omega.$$

In the above equations, a, b 6= 0, c 6= 0 and λ are known constants, f(t), g(u(t)) and h(u(t)) are given linear or nonlinear functions depending upon the particular problem discussed. In the next section, the Laplace decomposition method is described for the problem. In the ensuing section a set of three test problems are worked out. The last section includes concluding remarks.

4. AN ILLUSTRATION FOR THE APPLICATION OF THE METHODS FOR AN ORDINARY DIFFERENTIAL EQUATION (ODE)

A natural application of Newton's second law of motion to a simple problem of finding position of a damped harmonic oscillator as a function of time subjected to an external force when initial position and initial velocity are given can be formulated mathematically as an initial value problem for a second order ordinary differential equation. Description of physical quantities:

$$t \longrightarrow \text{time variable, } t \geq 0.$$

$$x(t) \longrightarrow \text{position or displacement of the oscillator, a bounded and infinitely differentiable function of } t.$$

$$x'(t) \longrightarrow \text{velocity of the oscillator.}$$

$$x''(t) \longrightarrow \text{acceleration of the oscillator.}$$

$$m \longrightarrow \text{mass of the oscillator.}$$

$$-2mx \longrightarrow \text{restoring force acting on the oscillator.}$$

$$-2mx'(t) \longrightarrow \text{damped force acting on the oscillator.}$$

$$me^{-t} \longrightarrow \text{external force acting on the oscillator.}$$

$$x(0) = 1 \longrightarrow \text{initial displacement.}$$

$$x'(0) = 0 \longrightarrow \text{initial velocity.}$$

Then the initial value problem is

$$mx''(t) = -2mx'(t) - 2mx(t) + me^{-t}, \quad t \geq 0$$

$$x(0) = 1, \; x'(0) = 0$$

$$\text{or} \quad x''(t) = -2x'(t) - 2x(t) + e^{-t}, \quad t \geq 0$$

$$x(0) = 1, \; x'(0) = 0.$$

Since coefficients of x 0 (t) as well as x(t) and e −t are continuous functions, by Picard's theorem, the initial value problem has a unique solution. First, it is easy to reduce the problem to a simpler problem without damping term as follows:

Put y(t) = e tx(t) in

$$x''(t) + 2x'(t) + 2x(t) = e^{-t}, \qquad x(0) = 1, \quad x'(0) = 0$$
$$\Leftrightarrow \frac{d^2y}{dt^2} + y = 1, \quad y(0) = 1, \qquad y'(0) = 1.$$

Application of Adomian Decomposition Method to ODE

Let us apply Adomian decomposition series [1]-[3] $y(t) = 1 + \sum_{n=1}^{\infty} y_n(t)$ on both sides of $y''(t) + y(t) = 1$ with the initial condition $y(0) = 1 \text{ and } y'(0) = 1,$ to get

$$\sum_{n=1}^{\infty} \frac{d^2y_n}{dt^2} + \sum_{n=1}^{\infty} y_n(t) = 0 \quad \text{or} \quad \frac{d^2y_1}{dt^2} + \sum_{n=2}^{\infty} \frac{d^2y_n}{dt^2} = 0 - \sum_{n=1}^{\infty} y_n(t).$$

A simple iteration,

$$\begin{aligned} \frac{d^2y_1}{dt^2} &= 0, \quad y_1(0) = 0, \ y_1'(0) = 1. \\ \frac{d^2y_k}{dt^2} &= -y_{k-1}, \quad y_k(0) = 0 = y_k'(0), \quad k = 2, 3, 4, \ldots \end{aligned}$$

will readily yield

$$y_1 = t, \ y_2 = -\frac{t^3}{3!}, \ \ldots, \ y_k = (-1)^{k-1} \frac{t^{2k-1}}{(2k-1)!}, \ \ldots .$$

Hence $y(t) = 1 + t - \frac{t^3}{3!} + \cdots + (-1)^{k-1} \frac{t^{2k-1}}{(2k-1)!} + \cdots = 1 + \sin t$ and $x(t) = e^{-t}y(t) = e^{-t}(1 + \sin t)$ is the desired exact solution.

Application of Laplace Transform Method to ODE

Let us apply Laplace transform [11, 13] on both sides of

$$y''(t) + y(t) = 1,$$

with the initial condition y(0) = 1 and y 0 (0) = 1, to get

$$\begin{aligned} s^2 L\{y(t)\} - s - 1 + L\{y(t)\} &= \frac{1}{s} \\ L\{y(t)\} &= \frac{s+1}{s^2+1} + \frac{1}{s(s^2+1)} = \frac{1}{s} + \frac{1}{s^2+1} \\ y(t) &= 1 + \sin t. \end{aligned}$$

Hence we get the desired exact solution

$$x(t) = e^{-t} y(t) = e^{-t}(1 + \sin t).$$

Application of Laplace Decomposition Method to ODE

Let us apply decomposition series technique in the Laplace transform method, we take straightaway the equation

$$L\{y(t)\} = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s^2} L\{y(t)\}.$$

The Laplace decomposition series [5] can be taken as

$$\begin{aligned} L\{y(t)\} &= \frac{1}{s} + \sum_{n=1}^{\infty} L\{y_n(t)\} \\ \frac{1}{s} + \sum_{n=1}^{\infty} L\{y_n(t)\} &= \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s^2}\left[\frac{1}{s} + \sum_{n=1}^{\infty} L\{y_n(t)\}\right] \\ L\{y_1(t)\} + \sum_{n=2}^{\infty} L\{y_n(t)\} &= \frac{1}{s^2} - \frac{1}{s^2} \sum_{n=2}^{\infty} L\{y_{n-1}(t)\}. \end{aligned}$$

We can obtain an iteration to compute $L\{y_n(t)\}$ as follows :

$$\begin{aligned} L\{y_1(t)\} &= \frac{1}{s^2} \\ L\{y_2(t)\} &= -\frac{1}{s^2} L\{y_1(t)\} = -\frac{1}{s^4} \\ &\vdots \\ L\{y_n(t)\} &= -\frac{1}{s^2} L\{y_{n-1}(t)\} = \frac{(-1)^{n-1}}{s^{2n}} \\ &\vdots \end{aligned}$$

Hence by using Laplace decomposition series for $L\{y(t)\}$, we arrive at

$$\begin{aligned} L\{y(t)\} &= \frac{1}{s}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{s^{2n}}, \quad s>1 \\ &= \frac{1}{s}+\frac{1}{s^2+1} \\ y(t) &= L^{-1}\left\{\frac{1}{s}\right\}+L^{-1}\left\{\frac{1}{s^2+1}\right\}=1+\sin t \end{aligned}$$

and hence the desired solution is $x(t)=e^{-t}y(t)=e^{-t}(1+\sin t)$.

Application of Laplace Transform Method to DDE [10]

Now multiplying both sides of (1.1.1) by e −st and integrate between ω and ∞, we obtain

$$\begin{aligned} \int_{\omega}^{\infty} y''(t)\, e^{-st}dt+\int_{\omega}^{\infty} y(t-\omega)\, e^{-st}dt &= \int_{\omega}^{\infty} 1\cdot e^{-st}dt \\ \int_{0}^{\infty} y''(t)e^{-st}dt+e^{-\omega s}\int_{0}^{\infty} y(t)e^{-st}dt &= \frac{e^{-\omega s}}{s} \\ L\{y''(t)\}+e^{-\omega s}L\{y(t)\} &= \frac{e^{-\omega s}}{s} \\ s^2L\{y(t)\}-s-1+e^{-\omega s}L\{y(t)\} &= \frac{e^{-\omega s}}{s} \end{aligned} \quad (1.1.3)$$

In step two, we have used $y''(t)=0, \quad 0\le t\le \omega$ and in step four, we have used $y(0)=1,\ y'(0)=1.$

$$\left(1+\frac{e^{-\omega s}}{s^2}\right)L\{y(t)\} = \frac{1}{s}+\frac{1}{s^2}+\frac{e^{-\omega s}}{s^3}$$

$$\begin{aligned} L\{y(t)\} &= \frac{1}{s}+\frac{1}{s^2}\frac{1}{\left(1+\frac{e^{-\omega s}}{s^2}\right)} \\ &= \frac{1}{s}+\frac{1}{s^2}+\sum_{n=1}^{\infty}(-1)^n\frac{e^{-n\omega s}}{s^{2n+2}}. \end{aligned}$$

Now by applying inverse Laplace transform, we get

$$y(t) = 1+t+\sum_{n=1}^{\infty}(-1)^n\frac{(t-n\omega)^{2n+1}}{(2n+1)!}e(t-n\omega). \quad (1.1.4)$$

Hence the exact solution in each interval is given by

$$y(t) = 1 + t + \sum_{n=1}^{N}(-1)^n \frac{(t-n\omega)^{2n+1}}{(2n+1)!}, \quad N\omega \le t \le (N+1)\omega,$$
$$N = 1, 2, 3, \dots \quad (1.1.5)$$

In (1.1.4), when we allow $\omega \to 0$, we get back the exact solution of the ODE, namely, $y(t) = 1 + \sin(t)$

5. SOME BASIC DEFINITIONS AND THEORETICAL RESULTS ON DIFFERENTIAL-DIFFERENCE EQUATIONS

We quote two basic definitions given in [10] :

Definition 1.2.1. By a differential-difference equation, we mean an equation in an unknown function and certain of its derivatives, evaluated at arguments which differ by any of a fixed number of values. The differential order of an equation is the order of the highest derivative appearing and by the difference order one less than the number of distinct arguments appearing in the equation. The general form of a linear differential-difference equation with constant coefficient of differential order n and difference of order m is

$$\sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij} u^{(j)}(t-\omega_i) = f(t), \quad (1.2.1)$$

where m and n are positive integers, $0 = \omega_0 < \omega_1 < \cdots < \omega_m$ and a_{ij} are real constants and f(t) is given real valued function defined for t > 0.

Definition 1.2.2. The set of all real functions having k continuous derivatives on an open interval $t_1 < t < t_2$ is denoted by $C^k(t_1, t_2)$. If f is a member of this set, that is $f \in C^k(t_1, t_2)$ for every $t_2 > t_1$, then $f \in C^k(t_1, \infty)$.

We quote two standard theorems on differential-difference equations of order (1, 1) with an initial interval condition [10] for existence of solution, continuity of the derivatives of the solution and applicability of Laplace transform method:

Theorem 1.2.1. Suppose that f is of class C^1 on $[0, \infty)$ and that g is of class C^0 on $[0, \omega]$. Then there exists one and only one function for $t \ge 0$ which is continuous for $t \ge 0$, which satisfies $u(t) = g(t)$ for $0 \le t \le \omega$ and which satisfies the equation in

$$a_0 u'(t) + b_0 u(t) + b_1 u(t-\omega) = f(t), \quad t > \omega. \quad (1.2.2)$$

Moreover, this function u is of class C^1 on (ω, ∞) and of class C^2 on $(2\omega, \infty)$. If g is of class C^1 on $[0, \omega]$, u' is continuous at ω if and only if

$$a_0 g'(\omega - 0) + b_0 g(\omega) + b_1 g(0) = f(\omega). \quad (1.2.3)$$

If g is of class C^2 on $[0, \omega]$, u'' is continuous at 2ω if either (1.2.3) holds or else $b_1 = 0$ and only in these cases.

Theorem 1.2.2. Let u(t) be a solution of the equation

$$L(u) = a_0u'(t) + b_0u(t) + b_1u(t-\omega) = f(t) \quad (1.2.4)$$

which is of class C 1 on [0,∞). Suppose that f is of class C 0 on [0,∞) and that

$$|f(t)| \leq c_1 e^{c_2 t}, \qquad t \geq 0, \quad (1.2.5)$$

where c1 and c2 are positive constants. Let $m = \max_{0 \leq t \leq \omega} |u(t)|.$ Then there are positive constants c3 and c4, depending only on c2 and the coefficients in (1.2.4), such that

$$|u(t)| \leq c_3(c_1 + m)\, e^{c_4 t}, \qquad t \geq 0.$$

For higher order equations, a similar theory on system of differential-difference equations with differential of order one is applied.

6. CONCLUSION

Many fundamental laws of physics and chemistry can be formulated as differential equations. In biology and economics, differential equations are used to model the behavior of complex systems. The mathematical theory of differential equations first developed together with the sciences where the equations had originated and where the results found application. However, diverse problems, sometimes originating in quite distinct scientific fields, may give rise to identical differential equations. Whenever this happens, mathematical theory behind the equations can be viewed as a unifying principle behind diverse phenomena. As an example, consider the propagation of light and sound in the atmosphere and of waves on the surface of a pond. All of them may be described by the same second-order partial differential equation, the wave equation, which allows us to think of light and sound as forms of waves, much like familiar waves in the water. Conduction of heat, the theory of which was developed by Joseph Fourier, is governed by another second-order partial differential equation, the heat equation. It turns out that many diffusion processes, while seemingly different, are described by the same equation; the Black–Scholes equation in finance is, for instance, related to the heat equation.

7. REFERENCES

1. M. Braun, Differential Equations and Their Applications, Springer-Verlag, New York, 1993.
2. R. Broson, Differential Equations, Tata McGraw-Hill, New Delhi, 2011.
3. W. H. Chen and Z. Y. Lu, An algorithm for Adomian decomposition method, Applied Mathematics and Computation, 159 (2004), 221-235.
4. C. Chicone, Ordinary Differential Equations with Applications, Springer, New Delhi, 2006.
5. E. A. Coddington, An Introduction to Ordinary Differential Equations, Dover, New York, 1989.
6. E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, Tata McGraw Hill, New York, 1955.
7. P. J. Collins, Differential and Integral Equation, Oxford University Press, New York, 2006.
8. C. Corduneanu, Integral Equations and Applications, Cambridge University Press, England, 1991.
9. H. T. Davis, Introduction to Nonlinear Differential and Integral Equations, Dover Publications, New York, 1962.
10. S. G. Deo, V. Lakshmikantham and V. Raghavendra, Ordinary Differential Equations, Tata McGraw Hill, New Delhi, 2010.
11. G. Doetsch, Introduction to the Theory and Application of the Laplace Transformation, Springer-Verlag, Berlin, 1974.
12. R. D. Driver, Ordinary and Delay Differential Equations, Springer-Verlag, New York, 1977.
13. L. E. El'sgol'ts and S. B. Norkin, Introduction to the Theory and Application of Differential Equations with Deviating Arguments, Academic Press, New York, 1973.

14. A. Ghorbani and J. S. Nadjafi, He's Homotopy perturbation method for calculating Adomian polynomials, International Journal of Nonlinear Sciences and Numerical Simulation, 8(2) (2007), 229-232.
15. A. M. Golberg, Solution Methods for Integral Equation: Theory and Applications, Plenum Press, NewYork, 1979.
16. M. A. Gondal, M. Khan and K. Omrani, A new Analytical approach to solve Magnetohydrodynamics flow over a nonlinear porous stretching sheet by Laplace Pade decomposition method, Results in Mathematics, 63 (2013), 289-301.