A Study on Numerical and Analytical Methods for Solving Ordinary Differential Equations

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Abstract

Differential Equations are among the most important Mathematical tools used in creating models in the science, engineering, economics, mathematics, physics, aeronautics, astronomy, dynamics, biology, chemistry, medicine, environmental sciences, social sciences, banking and many other areas A differential equation that has only one independent variable is called an Ordinary Differential Equation (ODE), and all derivatives in it are taken with respect to that variable. Most often, the variable is time, t; although, I will use x in this paper as the independent variable. The differential equation where the unknown function depends on two or more variables is referred to as Partial Differential Equations (PDE). In this paper, researcher presents the basic and commonly used numerical and analytical methods of solving ordinary differential equations. The results of these efforts are shown in the last sections of this chapter. We present theorems describing the existence and uniqueness of solutions to a wide class of first order differential equations.

Keywords: *- Differential equation, analytical methods, numerical, independent, Ordinary Differential Equation variable.*

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1. INTRODUCTION

A differential equation is an equation, where the unknown is a function and both the function and its derivatives may appear in the equation. Differential equations are essential for a mathematical description of nature— they lie at the core of many physical theories. For example, let us just mention Newton's and Lagrange's equations for classical mechanics, Maxwell's equations for classical electromagnetism, Schr¨odinger's equation for quantum mechanics, and Einstein's equation for the general theory of gravitation. We now show what differential equations look like. This chapter focuses on the background of the study, statement of the problem, objectives and purpose of the study, significance of the study and the scope of study. It also highlights the key preliminaries in the study and outlines how the study will be carried out.

2. REVIEW OF LITERATURE

Sebastian Walcher, Xiang Zhang, (2021) We study some fundamental properties of autonomous polynomial differential equations over the quaternions H. After showing that every polynomial differential equation in R4 can be rewritten as a differential equation over H, we specialize the investigation to quaternionic differential equations with coefficients in C⊆H, which retain some properties of polynomial differential equations in C. We provide a detailed characterization of this class and discuss their stationary points and periodic orbits, including those at infinity. As an application of our theory we investigate the dynamics of generic quadratic systems of this type and of a special class of quaternionic Riccati differential equations with complex coefficients. Here we obtain a new type of Liouville–Arnold integrable system.

James Kirkwood, (2019) This chapter discusses a nonhomogeneous linear second-order ordinary differential equation, with given boundary conditions, by presenting the solution in terms of an integral. The chapter demonstrates three methods of constructing Green's functions—that is, by using the Dirac-delta function, variation of parameters, and eigenfunction expansions. A second method for constructing Green's function is based on the theorem that uses the technique called variation of parameters to find a particular solution to certain second-order differential equations. The chapter also discusses the Fredholm alternative and Green's function for the Laplacian in higher dimensions. Then, the fundamental solution or the free-space Green's function (the situation with no boundary conditions) for the negative Laplacian in two and three dimensions is derived in the chapter.

Rodica Luca,(2018) This is concerned with the existence, multiplicity, and nonexistence of positive solutions for some classes of systems of nonlinear Riemann–Liouville fractional differential equations with parameters or without parameters, subject to uncoupled Riemann–Stieltjes integral boundary conditions, and for which the nonlinearities are nonsingular or singular functions. A system of fractional equations with sign-changing nonlinearities and integral boundary conditions is also investigated. Some examples which support our main results are also given.

Johnny Henderson, (2016) This chapter is concerned with the existence, multiplicity, and nonexistence of positive solutions for some classes of systems of nonlinear second-order ordinary differential equations with parameters or without parameters, subject to Riemann–Stieltjes integral boundary conditions, and for which the nonlinearities are nonsingular or singular functions. Some examples which illustrate our main results are also presented.

Frank E. Harris, (2014) A second-order ODE with independent variable and dependent variable may be such that y only appears as the derivatives y' and y''. The ODE can then be converted into a first-order equation by the substitution $y' = u, \quad y'' = u'$. If our ODE was nonlinear, the equation (now first-order, and with dependent variable u) may be recognizable as one of the few nonlinear first-order ODEs for which a solution is known. But if our original ODE was linear, we now have a linear first-order equation which we know how to solve (for u (x), its dependent variable). We complete the solution process by integrating u to obtain y(x).

3. OVERVIEW OF DIFFERENTIAL EQUATIONS

A differential equation is an equation, where the unknown is a function and both the function and its derivatives may appear in the equation. Differential equations are essential for a mathematical description of nature— they lie at the core of many physical theories. For example, let us just mention Newton's and Lagrange's equations for classical mechanics, Maxwell's equations for classical electromagnetism, Schr¨odinger's equation for quantum mechanics, and Einstein's equation for the general theory of gravitation. We now show what differential equations look like.

Example 1

(a) Newton's law:

Mass times acceleration equals force, ma = f, where m is the particle mass, $a = d^2x/dt^2$ is the particle acceleration, and f is the force acting on the particle. Hence Newton's law is the differential equation

$$m\frac{d^2\boldsymbol{x}}{dt^2}(t) = \boldsymbol{f}\left(t, \boldsymbol{x}(t), \frac{d\boldsymbol{x}}{dt}(t)\right),$$

Where the unknown is x(t)—the position of the particle in space at the time t. As we see above, the force may depend on time, on the particle position in space, and on the particle velocity.

Remark: This is a second order Ordinary Differential Equation (ODE).

(b) The Heat Equation:

The temperature T in a solid material changes in time and in three space dimensions—labeled by x = (x, y, z)—according to the equation

$$\frac{\partial T}{\partial t}(t, \boldsymbol{x}) = k\left(\frac{\partial^2 T}{\partial x^2}(t, \boldsymbol{x}) + \frac{\partial^2 T}{\partial y^2}(t, \boldsymbol{x}) + \frac{\partial^2 T}{\partial z^2}(t, \boldsymbol{x})\right), \qquad k > 0,$$

where k is a positive constant representing thermal properties of the material.

Remark: This is a first order in time and second order in space PDE.

The equations in examples (a) is called ordinary differential equations (ODE)— the unknown function depends on a single independent variable, t. The equations in examples (b) is called partial differential equations (PDE)—the unknown function depends on two or more independent variables, t, x, y, and z, and their partial derivatives appear in the equations.

4. HISTORY OF ORDINARY DIFFERENTIAL EQUATIONS

The attempt to solve physical problems led gradually to mathematical models involving an equation in which a function and its derivatives play important roles. However, the theoretical development of this new branch of mathematics - Ordinary Differential Equations - has its origins rooted in a small number of mathematical problems. These problems and their solutions led to an independent discipline with the solution of such equations an end in itself.

In circa 1671, English physicist Isaac Newton wrote his then-unpublished The Method of Fluxions and Infinite Series (published in 1736), in which he classified first order differential equations, known to him as fluxional equations, into three classes, as follows (using modern notation):

Ordinary differential equations		Partial differential equations
Class 1	Class 2	Class 3
$\frac{dy}{dx} = f(x)$	$\frac{dy}{dx} = f(x,y)$	$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$

The first two classes contain only ordinary derivatives of one or more dependent variables, with respect to a single independent variable, and are known today as "ordinary differential equations"; the third classes involves partial derivatives of one dependent variable and today are called "partial differential equations".

The study of "differential equations", according to British mathematician Edward Ince, and other mathematics historians is said to have begun in 1675, when German mathematician Gottfried Leibniz wrote the following equation

$$\int x\,dx = \frac{1}{2}x^2.$$

In 1676, Newton solved his first differential equation. That same year, Leibniz introduced the term “differential equations” (aequatio differentialis, Latin) or to denote a relationship between the differentials dx and dy of two variables x and y.

In 1693, Leibniz solved his first differential equation and that same year Newton published the results of previous differential equation solution methods—a year that is said to mark the inception for the differential equations as a distinct field in mathematics.

Swiss mathematicians, brothers Jacob Bernoulli (1654-1705) and Johann Bernoulli (1667-1748), in Basel, Switzerland, were among the first interpreters of Leibniz' version of differential calculus. They were both critical of Newton's theories and maintained that Newton’s theory of fluxions was plagiarized from Leibniz' original theories, and went to great lengths, using differential calculus, to disprove Newton’s Principia, on account that the brothers could not accept the theory, which Newton had proven, that the earth and the planets rotate around the sun in elliptical orbits.

The first book on the subject of differential equations, supposedly, was Italian mathematician Gabriele Manfredi’s 1707 On the Construction of First-degree Differential Equations, written between 1701 and 1704, published in Latin. The book was largely based or themed on the views of the Leibniz and the Bernoulli brothers. Most of the publications on differential equations and partial differential equations, in the years to follow, in the 18th century, seemed to expand on the version developed by Leibniz, a methodology, employed by those as Leonhard Euler, Daniel Bernoulli, Joseph Lagrange, and Pierre Laplace.

5. NEED OF A SOLUTION TO A DIFFERENTIAL EQUATION

Differential equations can describe nearly all systems undergoing change. They are ubiquitous is science and engineering as well as economics, social science, biology, business, health care, etc. Many researchers and mathematicians have studied the nature of Differential Equations and many complicated systems that can be described quite precisely with mathematical expressions.

Solutions of ordinary differential equations (ODEs) are in general possible by different methods. The main methods of solving ordinary differential equations are analytical and numerical; all other approaches are subsets of these.

6. NUMERICAL METHODS OF SOLVING AN ORDINARY DIFFERENTIAL EQUATION

Numerical methods are those used to find numerical approximations to the solutions of ordinary differential equations (ODEs). Their use is also known as "numerical integration", although this term is sometimes taken to mean the computation of integrals. Although many of the differential equations which result from modeling real-world problems can be solved analytically, there are many others which cannot. In general, when the modeling leads to a linear differential equation, the prospects of obtaining an exact mathematical solution are good. However, non-linear differential equations present much greater difficulty and exact solutions can seldom be obtained. There is a need, therefore, for numerical methods that can provide approximate solutions to problems which would otherwise be intractable. The advent of powerful computers capable of performing calculations at very high speed has led to a rapid development in this area and there are now many numerical methods available. Thus numerical methods are useful because many differential equations in practice cannot be solved using symbolic computation ("analysis") and so for practical purposes, such as in engineering – a numeric approximation to the solution is often sufficient. The algorithms studied in this research can be used to compute such an approximation.

Consider the ODE below

$$\frac{dy}{dx} = f(x,y)$$

Subject to initial conditions $(x_0) = y_0$ and suppose the solution $y(x)$ for $x \geq x_0$ is required. Suppose the graph shown below represents the exact solution

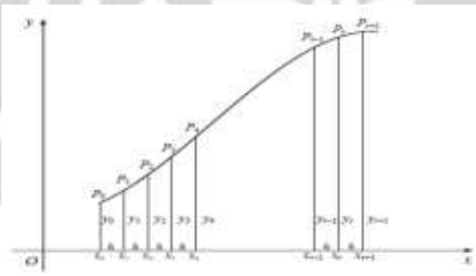

Figure 1: Figure of initial conditions

Let $x_0, x_1, x_2, \ldots$ … be points along the x-axis which are equally spaced a distance h apart and their corresponding y-values $y(x_0), y(x_1), y(x_2), ..$ be represented as $y_0, y_1, y_2, \cdots$ and let $P_0, P_1, P_2, \cdots$ be points on the curve with coordinates $(x_0, y_0), (x_1, y_1), (x_2, y_2) \cdots$ respectively. Therefore, in general, $y(x_r) = y_r$ where $x_r = x_0 + rh$ and P_r is a point with coordinate (, y_r).

- **Euler's method**

Consider figure 1 above, one simple way of obtaining an approximation to the value of y_1 is to assume that the part of the curve between P_0 and P_1 is a straight-line segment with gradient equal to the gradient of the curve at P_0. Since $d_y/d_x = (x, y)$, the gradient of the curve at P_0 is (x_0, y_0). Hence with this approximation,

$$\frac{y_1 - y_0}{h} = f(x_0, y_0)\,,$$

Giving

$$y_1 = y_0 + hf(x_0, y_0).$$

Using this approximation, we obtain the value of y at P_1, The process can be repeated, assuming that the part of the curve between P_1 and P_2 is a straight line segment with gradient equal to the value of $\frac{dy}{dx}$ at $P_{1.}$. This gives

$$y_2 = y_1 + hf(x_1, y_1).$$

More generally,

$$y_{r+1} = y_r + hf(x_r, y_r), \quad r = 0,1,2, \ldots.$$

This is Euler's formula. Successive calculation of values of y using this formula is known as Euler's method. It is clear from the nature of the linear approximation on which this method is based that the step length h needs to be fairly small to achieve reasonable accuracy.

- **Runge-Kutta Methods**

From Euler's formula,

$$y_{r+1} = y_r + hf(x_r, y_r), \quad r = 0,1,2, \ldots.$$

We denote $_1 = f(x_r , y_r)$

Therefore, going forth to the approximation to $x_r + \frac{1}{2}h$

$$k_2 = f(x_r + \frac{1}{2}h,\; y_r + \frac{1}{2}hk_1),$$

$$k_3 = f(x_r + \frac{1}{2}h,\; y_r + \frac{1}{2}hk_2),$$

$$k_4 = f(x_r + \frac{1}{2}h,\; y_r + \frac{1}{2}hk_3)\,.$$

Hence

$$y_{r+1} = y_r + \frac{h}{6}\{k_1 + 2k_2 + 2k_3 + 3k_4\}.$$

This gives a local truncation error proportional to h^5 and since $h < 1$ in most cases; it means the error is much less with the fourth order Runge-Kutta method.

Each of the , $i = 1,2, \ldots$ represents a k^{th} order Runge-Kutta method, the fourth order is the most stable and easy to implement.

7. ACCURACY OF THE EULER AND RUNGE KUTTA METHODS

The two main numerical methods studied in this research were the Euler method and the 4th Runge kutta method. Both methods are useful in obtaining approximate solutions to differential equations, in fact, they become even more useful in cases where the analytical solution does not exist. It is therefore appropriate to conclude with a note on the accuracy of these methods. When both methods were used to solve the same initial value problems, it was found that for the same step size, h, the error associated with Euler methods was quite larger than that obtained from the 4^{th} Runge Kutta method which agrees with the theory

8. CONCLUSION

The main purpose of this research study was to bring together the different analytical and numerical methods for solving ordinary differential equations. Emphasis was laid on first and second order equations although some of the methods discussed primarily apply even to higher order differential equations. Scientific computation was used in this study to ease work in doing repetitive calculations and avoid human errors. Euler and Runge kutta algorithms were implemented. Future researchers should also give more attention to finding solutions to systems of Ordinary differential equations and higher order differential equations using both the analytical methods, whenever possible and also Numerical methods.

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