A two-parameter third-order family of methods for solving nonlinear equations

Abdulwali azamsafi

Department of Mathematics, faculty of Science, parwan University.

Abstract

In this paper, we present a new two-parameter family of iterative methods for solving nonlinear equations which includes, as a particular case, the classical Potra and Pta´k third-order method. Per iteration the new methods require two evaluations of the function and one evaluation of its first derivative. It is shown that each family member is cubically convergent. Several examples are given to illustrate the performnce of the family members.

1. Introduction

We consider iterative methods to find a simple root α, i.e. $f(\alpha)=0$ *and* $f'(\alpha)\neq 0$ of a nonlinear equation $f(x)=0$ that uses f *and* f' but not the higher derivatives of f.

The best known iterative method for the calculation of α is Newton's method defined by

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)} \qquad (1)$$

where x_0 is an initial approximation sufficiently close to α. This method is quadratically convergent [1]. There exists a modification of Newton's method with third-order convergence due to Potra and Pta´k [2]

$$x_{n+1}=x_n-\frac{f(x_n)+f'\left(x_n-f(x_n)/f'(x_n)\right)}{f'(x_n)}, \qquad (2)$$

Some Newton-type methods with third-order convergence that do not require the computation of second derivatives have been developed [3–12]. To obtain some of those iterative methods the Adomian decomposi. tion method was applied in [3,4], He's homotopy perturbation method [5,6] and Liao's homotopy analysis

method [7]. Some of the other methods have been derived by considering different quadrature formulas for the computation of the integral arising from Newton's theorem:

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt \tag{3}$$

Weerakoon and Fernando [8] applied the rectangular and trapezoidal rules to the integral of (3) to rederive the Newton method and arrive at the cubically convergent method

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'\left(x_n - f(x_n)/f'(x_n)\right)}, \tag{4}$$

while Frontini and Sormani [9] obtained the cubically convergent method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\left(x_n - f(x_n)/2f'(x_n)\right)}, \tag{5}$$

by considering the midpoint rule.

In [10], Homeier derived the following cubically convergent iteration scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{+f'\left(x_n - f(x_n)/f'(x_n)\right)}\right), \tag{6}$$

by considering Newton's theorem for the inverse function $x = f(y)$ instead of $y = f(x)$.

Recently, Kou et al. in [11] considered Newton's theorem on a new interval of integration and arrived at the following cubically convergent iterative scheme

$$x_{n+1} = x_n - \frac{f\left(x_n + f(x_n)/f'(x_n)\right) - f(x_n)}{f'(x_n)}, \tag{7}$$

The aforementioned methods require three functional evaluations of the given function and its first derivative, but no evaluations of the second or higher derivatives. Finding the iterative methods with third-order converb gence, not requiring the computation of second derivatives is important and interesting from the practical point of view and becomes active now.

In this paper, we present a new two-parameter family of modified Newton's methods that does not require the computation of second-order derivatives of the function. Derivation of the family is based on finding a correction term for the second substep in the two-substep Newton method, which will be described in the folc lowing section. We prove that each family member is a third-order convergent. In particular, we show that the Potra and Pta´k third-order method can be obtained as a special case of the new family. Finally, the compark ison with other third-order methods is given to illustrate the performance of the presented methods.

2. Iterative methods and convergence analysis

We consider the two-substep Newton method given by

$$z_n = z_n - \frac{f(z_n)}{f'(z_n)} \tag{8}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} \tag{9}$$

Our aim is to find a correction term for the second substep (9) that will yield a third-order method. To do this, first consider fitting the function f(x) around the point $(x_n, f(x_n))$ with the third-degree polynomial

$$g(x) = ax^3 + bx^2 + cx + d \tag{10}$$

Imposing the tangency condition at the nth iterate x_n

$$g'(x_n) = f'(x_n) \tag{11}$$

On to (10), we have

$$c = f'(x_n) - 3a x_n^2 - 2bx_n \tag{12}$$

Thereby obtaining the first derivative of the approximating polynomial

$$g'(x) = 3ax^2 + 2bx + f'(x_n) - 3ax_n^2 - 2bx_n \tag{13}$$

Now, when z_n is defined by (8) we approximate $f'(z_n)$ as

$$f'(z_n) \approx g'(z_n) = \frac{f'^2(x_n) - (6ax_n + 2b) f(x_n) + 3af^2(x_n)}{f'(x_n)} \tag{14}$$

Using (14) in (9) we obtain the new two-parameter family of methods

$$z_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{15}$$

$$x_{n+1} = z_n - \frac{f(z_n) f'(z_n)}{f'^2(x_n) + (\lambda - 2\mu x_n) f(x_n) + \mu f^2(x_n)}, \tag{16}$$

Where $\lambda = -2b \in \mathbb{R}$, $\mu = 3a \in \mathbb{R}$ for the methods defined by (16), we have

Theorem 2.1. Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $f : I \to \mathbb{R}$ for an open interval I .if x_0 is sufficiently close to α ,then the order of convergence of the methods defined by (16) is three, and it then satisfies the error equation

$$e_{n+1} = c_2\left(2c_2 + \frac{\lambda - 2\mu\alpha}{f'(\alpha)}\right)e_n^3 + o\left(e_n^4\right) \qquad (17)$$

Where $e_n = x_n - \alpha, c_k = f^{(k)}(\alpha)/k!\,f'(\alpha)\, and\ \lambda, \mu \in \mathbb{R}$.

Proof: Let α be a simple zero of f . Using taylor expansion around $x_n = \alpha$ and taking into account $f(\alpha) = 0$,we have

$$f(x_n) = f'(\alpha)\left[e_n + c_2e_n^2 + o\left(e_n^3\right)\right], \qquad (18)$$

$$f'(x_n) = f'(\alpha)\left[I + 2c_2e_n + o\left(e_n^2\right)\right], \qquad (19)$$

Where $e_n = x_n - \alpha\ and\ c_k = \frac{1}{k!}f^{(k)}(\alpha)/f'(\alpha)$ $, k = 1, 2, 3...$ by a simple calculation, we get

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2\left(c_2^2 - c_3\right)e_n^3 + o\left(e_n^4\right), \qquad (20)$$

So that

$$z_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2e_n^2 + 2\left(c_2^2 - c_3\right)e_n^3 + o\left(e_n^4\right), \qquad (21)$$

Hence

$$f(z_n) = f'(\alpha)\left[c_2e_n^2 + 2\left(c_2^2 - c_3\right)e_n^3 + o\left(e_n^4\right)\right], \qquad (22)$$

From (18), (19) and (22)

$$f(z_n)\,f'(x_n) = f'^2(\alpha)\left[c_2e_n^2 + 2c_3e_n^3 + o\left(e_n^4\right)\right], \qquad (23)$$

And

$$f'^2(x_n) + (\lambda - 2\mu x_n)\,f(x_n) + \mu f^2(x_n) = f'(\alpha)\left[f'(\alpha) + \left(4c_2\,f'(\alpha) + \lambda - 2\mu\alpha\right)e_n + o\left(e_n^2\right)\right], \qquad (24)$$

Hence

$$\frac{f(z_n)\,f'(x_n)}{f'^2(x_n) + (\lambda - 2\mu x_n)\,f(x_n) + \mu f^2(x_n)} = c_2e_n^2 - \left(4c_2^2 + \frac{\lambda - 2\mu\alpha}{f'(\alpha)}c_2 - 2c_3\right)e_n^3 + o\left(e_n^4\right) \qquad (25)$$

It then follows from (21) and (25) that

$$x_{n+1} = z_n - \frac{f(z_n) f'(x_n)}{f'^2(x_n) + (\lambda - 2\mu x_n) f(x_n) + \mu f^2(x_n)} = \alpha + c_2\left(2c_2 + \frac{\lambda - 2\mu\alpha}{f'(\alpha)}\right)e_n^3 + o\left(e_n^4\right) \qquad (26)$$

Since $e_{n+1} = x_{n+1} - \alpha$,this shows that the iterative methods defined by (16) have third-order convergence indeh pendent of any real values of k and 1. This completes the proof.

The family (16) includes, as particular cases, the following ones:

For $\lambda = 0, \mu = 0$ we obtain the Potra and Pta´k third-order method (2):

$$x_{n+1} = z_n - \frac{f(x_n)}{f'(x_n)} \qquad (27)$$

Where z_n is defined by (15).

for $\lambda = 1, \mu = 0$, we obtain new third-order method:

$$x_{n+1} = z_n - \frac{f(x_n) f'(x_n)}{f'^2(x_n) + f(x_n)}, \qquad (28)$$

Where z_n is defined by (15).

for $\lambda = 0, \mu = 1$, we obtain another new third-order method:

$$x_{n+1} = z_n - \frac{f(x_n) f'(x_n)}{f'^2(x_n) - 2x_n f(x_n) + f^2(x_n)}, \qquad (29)$$

3. Numerical examples

All computations were done using MAPLE using 64 digit floating point arithmetics $(Digits := 64)$. We accept an approximate solution rather than the exact root, depending on the precision (ε) of the computer. We use the following stopping criteria for computer programs: (i) $|x_{n+1} - x_n| < \varepsilon$, (ii) $|f(x_{n+1})| < \varepsilon$, and so, when the stopping criterion is satisfied, xnþ1 is taken as the exact root a computed. We used $\varepsilon = 10^{-15}$.

We present some numerical test results for various cubically convergent iterative schemes in Table 1. Com. pared were the Newton method (NM), the method of Weerakoon and Fernando (4) (WF), the method derived from midpoint rule (5) (MP), the method of Homeier (6) (HM), the method of Kou et al. (7) (KM), and the methods (28) (CM1) and (29) (CM2) introduced in this paper. We remark that chosen for comparison are only the methods which do not require the computation of second or higher derivatives of the function to carry out iterations. We used the following test functions:

$$f_1(x) = x^2 + 4x^2 - 10$$
$$f_2(x) = \sin^2 x - x^2 + 1,$$
$$f_3(x) = x^2 - e^x - 3x + 2,$$
$$f_4(x) = \cos x - x,$$
$$f_5(x) = (x-1)^2 - 1,$$
$$f_6(x) = \sin x - x/2,$$
$$f_7(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5,$$

As convergence criterion, it was required that the distance of two consecutive approximations δ for the zero was less than 10^{-15}. Also displayed are the number of iterations to approximate the zero (IT), the approximate zero x_*, and the value $f(x_*)$. Note that the approximate zeroes were displayed only up to the 28th decimal places, so it making all looking the same though they may in fact differ.
It is clear from Table 1 that the proposed methods in this work show at least equal performance as comt pared with the other known methods of the same order. It is also seen from these numerical experiments that

Table 1

Comparison of various cubically convergent iterative methods and the newton method

	IT	x_*	$f(x_*)$	δ
$f_1, x_0 = 1.27$				
NM	5	1.3652300134140968457608068290	2.72e-41	**1.83e-21**
WF	4	1.3652300134140968457608068290	0	**3.0e -35**
MP	4	1.3652300134140968457608068290	0	**2.60e-35**
HM	3	1.3652300134140968457608068290	-4.45e-48	**2.07e-16**
KM	4	1.3652300134140968457608068290	0	**1.77e-33**
CM1	4	1.3652300134140968457608068290	0	**2.26e-31**
CM2	4	1.3652300134140968457608068290	0	**2.38e-33**
$f_2, x_0 = 3.5$				
NM	7	1.4044916482153412260350868178	-3.03e-43	**3.95e-22**
WF	5	1.4044916482153412260350868178	-2.0e-63	**2.12e-30**
MP	5	1.4044916482153412260350868178	-4.56e-61	**6.76e-21**
HM	5	1.4044916482153412260350868178	-2.0 e-63	**9.30e-33**
KM	5	1.4044916482153412260350868178	1.18e-45	**7.67e-28**
CM1	5	1.4044916482153412260350868178	1.3 e-63	**1.47e-28**
CM2	6	1.4044916482153412260350868178	-2.0e-63	**2.47e-40**
$f_3, x_0 = 0$				
NM	5	0.25753028543986076045536730494	-1.90e-35	**7.16e-18**
WF	4	0.25753028543986076045536730494	0	**1.97e-34**
MP	3	0.25753028543986076045536730494	0	**4.0 e-27**

HM	4	0.25753028543986076045536730494	0	**4.0 e-29**
KM	4	0.25753028543986076045536730494	-6.07e-57	**2.50e-19**
CM1	4	0.25753028543986076045536730494	0	**8.11e-29**
CM2	4	0.25753028543986076045536730494	-6.17e-58	**1.08e-19**
$f_4, x_0 = 1.2$				
NM	5	0.73908513321516064165531208767	-1.90e-35	**7.16e-18**
WF	4	0.73908513321516064165531208767	0	**1.97e-34**
MP	4	0.73908513321516064165531208767	0	**2.72e-27**
HM	4	0.73908513321516064165531208767	0	**4.0 e-29**
KM	4	0.73908513321516064165531208767	-6.07e-57	**2.50e-19**
CM1	4	0.73908513321516064165531208767	0	**8.11e-29**
CM2	4	0.73908513321516064165531208767	-6.17e-58	**1.08e-19**
$f_5, x_0 = 1.8$				
NM	6	2	2.87e-41	**3.09e-21**
WF	4	2	-4.01e-49	**4.86e-17**
MP	4	2	-7.98e-54	**1.43e-18**
HM	4	2	0	**6.52e-36**
KM	4	2	-1.56e-45	**7.30e-16**
CM1	5	2	0	**1.33e-23**
CM2	4	2	0	**3.08e-23**
$f_6, x_0 = 13$				
NM		Divergent		
WF	6	1.89549426703398094714403573381	1.63e-60	**1.87e-20**
MP	5	1.89549426703398094714403573381	-3.0 e-64	**2.93e-28**
HM		Divergent		
KM		Divergent		
CM1	7	1.89549426703398094714403573381	-9.85e-58	**9.56e-20**
CM2	51	1.89549426703398094714403573381	-9.79e-57	**1.53e-19**
$f_7, x_0 = -2$				
NM	9	-1.20764782713091892700941675840	-2.27e-40	**2.73e-21**
WF	7	-1.20764782713091892700941675840	-4.0 e-63	**3.11e-44**
MP	6	-1.20764782713091892700941675840	-4.0 e-63	**2.12e-23**
HM	6	-1.20764782713091892700941675840	-4.0 e-63	**2.57e-32**
KM	6	-1.20764782713091892700941675840	-4.0 e-63	**8.87e-34**
CM1	7	-1.20764782713091892700941675840	-4.0 e-63	**2.31e-35**
CM2	7	-1.20764782713091892700941675840	-4.0 e-63	**2.25e-34**

in almost all of the cases the presented methods appear to be as robust as compared other methods. The most important characteristic of the proposed methods is that they do not require to compute second or higher derivatives of the function to carry out iterations.

4. Conclusion

In this paper, we presented a new two-parameter family of modified Newton's methods which includes, as a particular case, the Potra and Pta´k third-order method. Per iteration the new methods require two evaluations of the function and one evaluation of its first derivative. We have shown that each family member is cubically convergent, and observed from numerical examples that the proposed methods show at least equal perforc mance as compared with the other known methods of the same order.

References

[1] A.M. Ostrowski, Solution of Equations in Euclidean and Banach space, Academic Press, New York, 1973.

[2] F.A. Potra, V. Pta´k, Nondiscrete induction and iterative processes, Research Notes in Mathematics, vol. 103, Pitman, Boston, 1984.

[3] S. Abbasbandy, Improving Newton–Raphson method for nonlinear equations by modified Adomian decomposition method, Appl. Math. Comput. 145 (2003) 887–893.

[4] C. Chun, Iterative methods improving Newton's method by the decomposition method, Comput. Math. Appl. 50 (2005) 1559–1568. [5] J.H. He,Newton-likeiteration methodsfor solvingalgebraicequations,Commun. NonlinearSci.Numer. Simul. 3 (2)(1998)106–109.

[6] S. Abbasbandy, Modified homotopy perturbation method for nonlinear equations and comparison with Adomian decomposition method, Appl. Math. Comput. 172 (2006) 431–438.

[7] C. Chun, Construction of Newton-like iteration methods for solving nonlinear equations, Numer. Math. 104 (3) (2006) 297–315.

[8] S. Weerakoon, G.I. Fernando, A variant of Newton's method with accelerated third-order convergence, Appl. Math. Lett. 17 (2000) 87–93.

[9] M. Frontini, E. Sormani, Some variants of Newton's method with third-order convergence, J. Comput. Appl. Math. 140 (2003) 419– 426.

[10] H.H.H. Homeier, On Newton-type methods with cubic convergence, J. Comput. Appl. Math. 176 (2005) 425–432.

[11] J. Kou, Y. Li, X. Wang, A modification of Newton method with third-order convergence, Appl. Math. Comput. doi:10.1016/ j.amc.2006.01.076.

[12] J.F. Traub, Iterative Methods for the Solution of Equations, Chelsea Publishing Company, New York, 1977.