Approximation of continuous real valued functions

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ABSTRACT

The aim of the article is to trace with some details the history of approximation of functions in \( C[a,b] \) beginning with Weierstrass and the constructive theory initiated by Bernstein.

Key words: Supremum norm, metric space, Banach space, sequence of polynomials, normed linear space.

Introduction:

Weierstrass in 1985 showed that a continuous real function defined on a compact interval can be approximated uniformly to any desired extent by a polynomial. Bernstein later showed that knowing the desired nearness of the polynomial to a function one could construct it. If \([a,b]\) is the compact interval and \(C[a,b]\) is the space of all continues functions on \([a,b]\) and \(P\) the space of all polynomials with real coefficients, then \(P\) is a dense linear subspace of the Banach space \(C[a,b]\) with the norm

\[
\| f \| = \sup_{a \leq x \leq b} |f(x)|, f \in C[a,b]
\]

In other words, given an \(f \in C[a,b]\), there exists for every \(\varepsilon > 0\), a \(p \in P\) which is near to \(f\) by less than \(\varepsilon\).

Weierstrass theorem:

Let \(f\) be in \(C[a,b]\). Then given \(\varepsilon > 0\), there exist a polynomial \(p\) such that

\[
|f(x) - p(x)| < \varepsilon, (a \leq x \leq b) \quad \text{--------} \quad (1)
\]

Remark:

\(C[a,b]\) is a complete metric space with metric \(\rho\) defined by

\[
\rho(f, g) = \|f - g\|, \quad (f, g \in C[a,b]) \quad \text{--------} \quad (2)
\]

Where \(\|\cdot\|\) is called the supremum norm of \(f\) defined by

\[
\| f \| = \max_{a \leq x \leq b} |f(x)| \quad \text{--------} \quad (3)
\]

(1) Is equivalent to the statement

There exists a sequence \(\{p_n\} \in P\) such that \(p_n\) converges to \(f\) uniformly on \([a,b]\).

(i.e) If \(p_n\) is chosen such that

\[
|f(x) - p_n(x)| < \frac{\varepsilon}{n}, (a \leq x \leq b) \quad \text{--------} \quad (4)
\]

where \(n\) is a positive integer such that \(\frac{\varepsilon}{n} < \varepsilon\).
\[ |f(x) - p_n(x)| < \frac{1}{n}, \ a \leq x \leq b, \text{ then } p_n \rightarrow f \text{ uniformly on } [a,b] \]

Conversely if \( \varepsilon > 0 \) is given, we need to choose \( n \) with \( \frac{1}{n} < \varepsilon \), a \( p_{n_0} \) with

\[ |f(x) - p_{n_0}(x)| < \frac{1}{n} \]

Thus the theorem can be restated depending on the context.

**Bernstein Proof of the theorem:**

Take \( a=0, b=1 \)

Let \([a,b]\) be any closed bounded interval.

Let \( f \in C[a,b] \)

Let \( g(x) = f[a+(b-a)x], 0 \leq x \leq 1 \)

Now \( g(0) = f(a), f(1) = f(b). \)

Clearly \( g \in C[0,1] \)

Hence there exists a polynomial \( Q \) such that \( |g(y) - Q(y)| < \varepsilon, 0 \leq y \leq 1 \)

If \( y = \frac{x-a}{b-a} \), then

\[ g(y) = g \left( \frac{x-a}{b-a} \right) \]

\[ = f \left( a + (b-a) \left( \frac{x-a}{b-a} \right) \right) = f(x) \]

Thus \( |f(x) - Q \left( \frac{x-a}{b-a} \right)| < \varepsilon, a \leq x \leq b \quad \rightarrow (4) \)

If \( p(x) = Q \left( \frac{x-a}{b-a} \right) \), then \( p \) is a polynomial as \( Q \) is a polynomial consider \( C[0,1] \).

For any \( f \in C[0,1] \) we define a sequence of polynomials \( B_n, n=1,2, \ldots \), as follows.

\[ B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f \left( \frac{k}{n} \right), 0 \leq x \leq 1 \quad \rightarrow (5) \]

Where \( \binom{n}{k} \) represents the number of \( k \) combinations out of \( n \).

\( B_n \) is called the \( n^{th} \) Bernstein polynomial for \( f \).

Given \( \varepsilon > 0 \), we show that there exists \( n \in N \) such that

\[ \| f - B_n \| < \varepsilon \quad (n \leq N) \]

For any \( p, q \in R \), by Binomial theorem
\[
\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = (p + q)^n, \quad n \in N \quad \rightarrow (6)
\]

Differentiate with respect to \( p \) we get
\[
\sum_{k=0}^{n} k \binom{n}{k} p^{k-1} q^{n-k} = n (p + q)^{n-1},
\]
\[
\sum_{k=0}^{n} \frac{k}{n} \binom{n}{k} p^{k-1} q^{n-k} = p (p + q)^{n-1}, \quad \rightarrow (7)
\]

Differentiate again with respect to \( p \),
\[
\sum_{k=0}^{n} \frac{k^2}{n(1-x)^{n-k}} \binom{n}{k} p^{k-1} q^{n-k} = \frac{p(1-x)^{n-k}}{n} + \frac{p}{n} (p + q)^{n-1} \quad \rightarrow (8)
\]

Sub \( p=x, \quad q=1-x, \quad 0<x<1 \)

Then (6), (7) & (8) becomes
\[
\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1
\]
\[
\sum_{k=0}^{n} \left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} = x
\]
\[
\sum_{k=0}^{n} \left(\frac{k^2}{n^2}\right) \binom{n}{k} x^k (1-x)^{n-k} = x^2 \left(1 - \frac{1}{n}\right) + \frac{x}{n} \quad \rightarrow (9)
\]

From (9) we have
\[
\sum_{k=0}^{n} \left(\frac{k}{n}-x\right)^2 \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^{n} \left(\frac{k^2}{n^2}\right) \binom{n}{k} x^k (1-x)^{n-k} - 2x \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} + x^2 \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}
\]
\[
= x^2 - \frac{x^2}{n} + \frac{x}{n} - 2x(x) + x^2
\]
\[
= \frac{x(1-x)}{n}
\]
\[
\sum_{k=0}^{n} \left(\frac{k}{n}-x\right)^2 \binom{n}{k} x^k (1-x)^{n-k} = \left(\frac{x(1-x)}{n}\right) \quad \rightarrow (10)
\]

Now \( f \in C[0,1] \) is uniformly continuous on the compact interval \([0,1]\).
Hence given \( \varepsilon>0 \), there exists \( \delta>0 \)
Such that \( |f(x)-f(y)|<\varepsilon/2 \), Whenever
\[|x-y|<\delta, \quad x, y \in [0,1]\]
Assuming $||f|| \neq 0$, we have $N$ such that
\[ \frac{1}{\sqrt{N}} < \delta \quad \Rightarrow (11) \]
\[ \frac{1}{\sqrt{N}} < \frac{\varepsilon}{4||f||} \quad \Rightarrow (12) \]

For fixed $x \in [0,1]$ we have
\[
f(x) - B_n(x) = \sum_{k=0}^{n} f\left(x - \frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}
\]
\[= \sum \text{'} + \sum \text{''} \quad \Rightarrow (13)\]

Where $\sum \text{'}$ is the sum over the values of $k$ such that
\[|\frac{k}{n} - x| < \frac{1}{\sqrt{n}} \quad \Rightarrow (14)\]

And $\sum \text{''}$ is the sum over the other values of $k$ for which
\[|\frac{k}{n} - x| < \frac{1}{\sqrt{n}}\]
\[(k-nx)^2 = n^2 |\frac{k}{n} - x|^2 \geq \sqrt{n^3}\]

Hence $|\sum \text{'}| = |\sum \text{'} f(x) - f\left(x - \frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}|
\leq \sum \text{'} |f(x)| + |f\left(x - \frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}|
\leq 2||f|| \sum \text{'} \binom{n}{k} x^k (1-x)^{n-k}|
\leq 2||f|| \sum_{k=0}^{n} (k-nx)^2 \binom{n}{k} x^k (1-x)^{n-k}\]
\[\leq 2||f|| \frac{\varepsilon}{\sqrt{n^3}} \sum_{k=0}^{n} (k-nx)^2 \binom{n}{k} x^k (1-x)^{n-k}\]
\[\leq 2||f|| \frac{\varepsilon}{\sqrt{n^3}} \sum_{k=0}^{n} (k-nx)^2 \binom{n}{k} x^k (1-x)^{n-k}\]
\[\Rightarrow |\sum \text{'}| \leq 2||f|| \frac{\varepsilon}{\sqrt{n^3}} (1-x) \leq \frac{2||f||}{\sqrt{n}}\]

If $n \geq N$ it follows from (12) that
\[\frac{1}{\sqrt{n}} < \frac{\varepsilon}{4||f||} \quad \text{and so}\]
\[ \left| \sum | \sum [ f(x) - f\left(\frac{k}{n}\right) ] \right| \leq \varepsilon \]

Moreover if \( n \geq N \) and if \( k \) refer(14) then by (11), \( \left| \frac{k}{n} - x \right| \leq \frac{1}{\sqrt{N}} \) and so

\[ |f(x) - f\left(\frac{k}{n}\right)| \leq \frac{\varepsilon}{2} \]

\[ \left| \sum | \sum [ (f(x) - f\left(\frac{k}{n}\right) ] \right| \frac{n}{k} x^k (1 - x)^{n-k} \]

\[ \leq \frac{\varepsilon}{2} \sum \left( \frac{n}{k} \right) x^k (1 - x)^{n-k} \]

\[ \leq \frac{\varepsilon}{2} \text{ by (9)} \]

Hence \( |f(x) - B_n(x)| \leq \varepsilon \]

\[ \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \]

**Conclusion:**

The aim of the article is to express how a classical notation has given rise to rich abstractions.

**References:**

1. Lorentz C.G. “Approximation of Functions “, Newyork, 1966