CHARACTERIZING COMPLETABLE AND SEPARABLE FUZZYMETRIC SPACE

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Abstract

In this paper we define characterizing of fuzzy metric space and set in a complete fuzzy metric space in a topologically complete fuzzy metric space and proved the subspace of a separable fuzzy metric space is also separable.

Key words

Fuzzy metric space, topologically fuzzy metric space, separable fuzzy metric space, completable.

Introduction

The concept of fuzzy sets and fuzzy logic was introduced by Professor Lofti A Zadeh in 1965. The success of research in fuzzy sets and fuzzy logic has been demonstrated in a variety of fields, such as artificial intelligence, computer science, control engineering, computer applications, robotics and many more. One of the most important problems in Fuzzy Topology is to obtain an appropriate concept of fuzzy metric space.

This problem has been investigated by many authors from di8erent points of view. In particular, George and Veeramani have introduced and studied a notion of fuzzy metric space with the help of continuous t-norms, which constitutes a slight but appealing modification of the one due to Kramosil and Michalek.

Definition 1.1

Let (X_1, d_1) and (X_2, d_2)be metric space and let $x = (x_1, x_2)$and $y = (y_1, y_2)$be arbitrary points in the product $X =(X_1 \times X_2)$. Define $d\,(x, y) = max\,\{d_1(x_1, y_1), d_2(x_2, y_2)\}$. Then $d\,(x, y) = max\,\{d_1(x_1, y_1), d_2(x_2, y_2)\}$ is a metric on X and (X, d) called the product of the metric spaces (X_1, d_1) and (X_2, d_2).

Definition 1.2

A fuzzy metric space is an ordered triple (X, M, $*$) such that X is a (nonempty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0,+\infty)$ satisfying the following conditions, for all x, y, $z \in X$, s,t$>$0:

(i) M(x,y,t)$>$ 0;

(ii) M(x,y,t)=1 if and only if x=y;

(iii) M(x,y,t)=M(y,x,t);

(iv) M(x,y,t)$*$M(y,z,s)$\leq$ M(x,z,t +s);

(v) M(x,y,-):(0 +∞)→(0,1] is continuous.

Definition 1.3

Let (X,M,$*$) and (Y,N,$\star$) be two fuzzy metric spaces. Then,

(a) A mapping f from X to Y is called an isometry if for each x,y∈X and each t> 0, M(x,y,t) =N(f(x),f(y),t).

(b) (X,M,$*$) and (Y,N, $\star$) are called isometric if there is an isometry from X onto Y.

Definition 1.4

Let (X,M,$*$) be a fuzzy metric space. A fuzzy metric completion of (X,M,$*$) is a complete fuzzy metric space (Y,N, $\star$) such that (X,M,$*$) is isometric to a dense subspace of Y.

Definition 1.5

A fuzzy metric space (X,M,$*$) is called completable if it admits a fuzzy metric completion.

Lemma 1.6

Let $(a_n)_n$be a Cauchy sequence in a fuzzy metric space (X,M,$*$) and let t>0. If for each k∈N there exists lim_n M(a_k,a_n ,t), then$\lim_k \lim_n$ M(a_k, a_n , t)=1.

Proof:

Choose an arbitrary ε∈(0,1). Since $(a_n)_n$ is a Cauchy sequence, there exists n_0∈N such that M(a_k,a_n ,t) >1−$\mathcal{E}$ for all k,n≥ n_0. Therefore, for each k≥ n_0,

lim_n M(a_k, a_n , t) $> 1 - \varepsilon$ and

Hence, $\lim_k \lim_n$ M(a_k, a_n , t) ≥1-ε.

We conclude that $\lim_k \lim_n M(a_k, a_n, t)$=1.

Lemma 1.7

Let (X,M,$*$) be a fuzzy metric space. Then, for each metric d on X compatible with M, the following hold:

(i) A sequence in X is Cauchy in (X,M,$*$) if and only if it is Cauchy in (X,d).

(ii) (X, M,$*$) is complete if and only if (X,d) is complete.

Proof :

Let d be a metric on X compatible with M. That every Cauchy sequence in (X, d) is a Cauchy sequence in (X,M,$*$), and, thus, if (X,M,$*$) is complete, then (X,d) is complete.

Conversely,

It is immediate to see that every Cauchy sequence in (X,M,$*$) is a Cauchy sequence in (X,U_M), and thus, in (X,d); hence, if (X,d) is complete, then (X,M,$*$) is complete.

Results 1.8

Every nested sequence of nonempty closed sets with metric diameter zero has nonempty intersection.

Results 1.9

Every separable metric satisfies the second condition of countability.

Definition 1.10

A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm(t-norm) if for all $a,b,c,d \in [0,1]$

i). $a * 1 = a$

ii). $a * b = b * a$ $(commutativity)$

iii). $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$

iv). $a * b * c = a * b * c$ $(associativity)$

Theorem 1.11

Let $(X_1, M_1, *)$ and $(X_2, M_2, *)$ be fuzzy metric spaces. We define $M((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) * M2(x_2, y_2, t)$ Then M is a complete fuzzy metric on $X_1 \times X_2$ if and only if $(X_1, M_1, *)$ and $(X_2, M_2, *)$ are complete.

Proof:

Suppose that $(X_1, M_1, *)$ and $(X_2, M_2, *)$ be fuzzy metric spaces. Let $\{a_n\}$ be a Cauchy sequence in

$X_1 \times X_2$.

Note that $a_n = (x_1^n, x_2^n)$ and $a_m = (x_1^m, x_2^m)$. Also, $(a_n, a_m,)$ converges to 1. This implies that $(x_1^n, x_2^n, (x_1^m, x_2^m), t)$ converges to 1 for each $t > 0$.

It follows that $M_1(x_1^n, x_1^m, t) * M_2(x_2^n, x_2^m, t)$ converges to 1 for each $t > 0$. Thus $M_1(x_1^n, x_1^m, t)$ converges to 1 and also $M_2(x_2^n, x_2^m, t)$ converges to 1.

Therefore $\{x_1^n\}$ is a Cauchy sequence in $(X_1, M_1, *)$ and $\{x_2^n\}$ is a Cauchy sequence in $(X_2, M_2, *)$.

Since $(X_1, M_1, *)$ and $(X_2, M_2, *)$ are complete fuzzy metric spaces, there exists $x_1 \in X_1$ and $x_2 \in X_2$ such that $M_1(x_1^n, x_1, t)$converges to 1 and $M_2(x_2^n, x_2, t)$ converges to 1 for each $t > 0$.

Let $a = (x_1, x_2)$. Then $a \in X_1 \times X_2$. It follows that $(a_n,, t)$ converges to 1 for each $t > 0$. This shows that $(X, M, *)$ is complete.

Conversely,

Suppose that $(X, M, *)$ is complete. We shall show that $(X, M_1, *)$ and $(X, M_2, *)$ are complete. Let $\{x_1^n\}$ and $\{x_2^n\}$ be Cauchy sequences in $(X, M_1, *)$ and $(X, M_2, *)$ respectively.

Thus $M_1(x_1^n, x_1^m, t)$ converges to 1 and $M_2(x_2^n, x_2^m, t)$ converges to 1 for each $t > 0$. It follows that $M(x_1^n, x_2^m, t) = M_1(x_1^n, x_1^m, t) * M_2(x_2^n, x_2^m, t)$ Converges to 1. Let $x^n = (x_1^n, x_2^m)$ in $X_1 \times X_2$ for

$n \geq 1$. Then $\{x^n\}$ is a Cauchy sequence in X.

Since $(X,, *)$ is complete, there exists $x \in X_1 \times X_2 = X$ such that $M(x_1^n, x, t)$ converges to 1. Since

$x \in X_1 \times X_2$, we may put $x = (x_1, x_2)$, $x_1 \in X_1$ and $x_2 \in X_2$. Clearly, $M_1(x_1^n, x_1,)$ converges to 1 and $M2(x_2^n, x_2, t)$ converges to 1.

Hence $(X, M_1, *)$ and $(X, M_2, *)$ are complete. This completes the proof.

Theorem 1.12

Every separable fuzzy metric space is second countable.

Proof:

Let $(X,,*)$ be the given separable fuzzy metric space. Let $A = \{a_n : n \in \mathbb{N}\}$, be a countable dense subset of X. Consider

$B = \{(aj,\frac{1}{k}, \frac{1}{k}):$j, k $\in \mathbb{N}\}$. Then B is countable. We claim that B is a base for the family of all open sets in X.

Let G be an arbitrary open set in X. Let $x \in G$, then there exists $r, > 0,0 < r < 1$, such that $B(x,r,t) \subset G$.

Since $r \in (0,1)$ we can find an $s \in (0,1)$ such that $(1-s) * (1-s) > (1-r)$. Choose m$\in \mathbb{N}$ such that $\frac{1}{m} < \min(s, \frac{t}{2})$.

Since A is dense in X, there exists $a_j \in A$ such that $a_j \in B(x, \frac{1}{m}, \frac{1}{m})$.

Now if $y \in B(a_j, \frac{1}{m}, \frac{1}{m})$

then,

$$
\begin{aligned}
M(x,y,t) &\geq M(x, a_j, \tfrac{t}{2}) * M(y, a_j, \tfrac{t}{2}) \\
&\geq M(x, a_j, \tfrac{1}{m}) * M(y, a_j, \tfrac{1}{m}) \\
&\geq (1-\tfrac{1}{m}) * (1-\tfrac{1}{m}) \\
&\geq (1-s) * (1-s) \\
&> 1-r.
\end{aligned}
$$

Thus $y \in (x,,t)$ and hence B is a basis. Hence the result.

Proposition 1.13

A subspace of a separable fuzzy metric space is separable.

Proof:

Let X be the given fuzzy metric space and Y be a subspace of X. Let $A = \{x_n, \in \mathbb{N}\}$ be a countable dense subset of X.

For arbitrary but fixed , $\in \mathbb{N}$, if there are points $x \in X$ such that $M(x_n,x, \frac{1}{k}) > 1 - \frac{1}{k}$, choose one of them and denote it by x_{n_k}.

Let $B = \{x_{n_k},, \in \mathbb{N}\}$, then B is countable. Now we claim that $Y \subset \bar{B}$. Let $y \in Y$. Given $r, > 0,0 < r < 1$, we can find $a,k \in \mathbb{N}$ such that $(1 - \frac{1}{K}) * (1 - \frac{1}{K}) > 1 - r$.

Since A is dense in X, there exists an $m \in \mathbb{N}$ such that $(x_m,y, \frac{1}{K}) > 1 - \frac{1}{K}$. But by definition of B, there exists $x_{m_k} \in A$ such that

$$M(x_{m_k}, x_m, \tfrac{1}{K}) > 1 - \tfrac{1}{K} .$$

Now

$M\ (x_{m_k}, y, t\ ) \geq M\ (x_{m_k}, x_m, \frac{t}{2}) * M\ (x_{m_k}, x_m, \frac{t}{2}\ )$

$\geq M(x_{m_k}, x_m, \frac{1}{K}\ ) * M(\ x_m, y, \frac{1}{K}\ )$

$\geq (1 - \frac{1}{K}\ ) * (1 - \frac{1}{K}\ )$

$= 1 - r.$

Thus $y \in \bar{B}$ and hence Y is separable.

Theorem 1.14

A necessary and sufficient condition that a fuzzy metric space $(X,, *)$ be complete is that every nested sequence of nonempty closed sets $\{F_n\}_{n=1}^{\infty}$ with fuzzy diameter zero has nonempty intersection.

Proof:

First suppose that the given condition is satisfied. We claim that $(X, M, *)$ is complete. Let $\{x_n\}$ be a Cauchy sequence in X.

Take $A_n = \{x_n, x_{n+1}, x_{n+2}, \ldots.\}$ and $F_n = \overline{A_n}$, then we claim that $\{F_n\}$ has fuzzy diameter zero. For given $s, t >$ 0,0 < s < 1, we can find an $r \in$ (0,1), such that ($1 - r) * (1 - r) * (\ 1 - r) > (1 - s)$.

Since $\{x_n\}$ is a Cauchy sequence, for r, t > 0,0 < r < 1, there exists $n_0 \in N$ such that

$M\ (x_n, x_m, \frac{t}{3}\ ) > 1 - r$

for all $n, m \geq n_0$. Therefore

$M\ (x,, \frac{t}{3}\ ) > 1 - r$

for all $x, y \in A_{n_0}$. Let $x, \in F_{n_0}$. Then there exists sequences $\{x_n^{'}\}$ and $\{y_n^{'}\}$ in A_{n_0}such that $x_n^{'}$ converges to x and $y_n^{'}$ converges to y. Hence $x_n^{'} \in B\ (x,, \frac{t}{3})$ and $y_n^{'} \in B(\ y, r, \frac{t}{3})$ for sufficiently large n. Now

$M(x,y,t) \geq M\ (x, x_n^{'}\ , \frac{t}{3}) * M(x_n^{'}\ , y_n^{'}, \frac{t}{3}) * M(y_n^{'}\ , y, \frac{t}{3})$

$> (1 - r) * (1 -\ \ ) * (1 - r)$

$> (1 - s)$

Therefore

$M\ (x,,) > 1 - s$

for all $x, y \in F_{n_0}$. Thus F_n has fuzzy diameter zero. Hence by hypothesis $\cap_{n=1}^{\infty} F_n$ is nonempty. Take

$x \in\ \cap_{n=1}^{\infty} F_n.$

Then for r, t > 0,0 < r < 1, there exists n_1 such that

$M\ (x_m,,) > 1 - r$

for all $n \geq n_1$. Therefore, for each t > 0, $(x_n,, t)$ converges to 1 as n tends to ∞. Hence $\{x_n\}$ converges x. Therefore $(X,, *)$ is a complete fuzzy metric space.

Conversely,

Suppose that $(X,, *)$ is fuzzy complete and $\{\ F_n\}_{n=1}^{\infty}$ is a nested sequence of nonempty closed sets with fuzzy diameter zero.

Let $x_n \in F_n$, = 1,2,3,…. Since $\{F_n\}$ has a diameter zero, for r, > 0,0 < r < 1, there exists $n_0 \in N$ such that M $(x,y,t) > 1 - r$ for all $x,y \in F_{n_0}$.

Therefore $M(x_n, x_m, \frac{t}{3}) > 1 - r$ for all $n,m \geq n_0$. Since $x_n \in F_n \subset F_{n_0}$.

and $x_m \in F_m \subset F_{n_0}$, $\{x_n\}$ is a Cauchy sequence. But $(X,,*)$ is a complete fuzzy metric space and hence $\{xn\}$ converges to x for some $x \in X$.

Now for each fixed n, $x_k \in F_n$ for all $k \geq n$. Therefore $\in \overline{F_n} = F_n$ for every n, and hence $x \in \cap_{n=1}^{\infty} F_n$. This completes our proof. x_m

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