DESIGN AND BALANCED CONTROL FOR DOUBLE PENDULUM SYSTEM: SIMULATION AND EXPERIMENT

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ABSTRACT

The cart-double pendulum system is one of the unstable experimental systems that fully converges the complex properties of nonlinear control problems. It represents a class of real world systems such as two-wheeled mobile robots, pendubots, missile launchers and many more. The problems associated with it are always challenging topics in the field of control systems which required a suitable and fast reaction controller. This paper presents a technique to control this system stabilizing at a upright position. Simulation and experimental results using Matlab/Simulink toolbox will show a better performance of the proposed controller under disturbance and change in mass.

Keyword: ***Cart-double pendulum;*** *Quadratic Optimal Regulator; Boundary value problem; balance control.*

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1. INTRODUCTION

The cart-double pendulum system has two equilibrium points [6], the stable point is at which the pendulum is pointing downwards and the unstable one is at which the pendulum is pointing upwards. The aim of designing a controller is to move and balance the pendulum from the stable equilibrium point to the unstable one. This is a challenging control problem because the system is highly unstable, nonlinear and underactuated. Different control agorithms are studied by many researchers, from classical PID controllers [14] to advanced controllers such as neural networks, [15] and optimal control using LQR controller [13], [17]. However, these algorithms are only in simulation.

The goal of this article is to design controllers to swing up and balance the pendulum from a pending position to the vertical upward point. Swinging up the pendulum can be achieved by using feedforward control [18]. At the vertical position, LQR controller is used to stabilize the pendulum. A switch is used to change controllers. This means, when the pendulum approaches a certain area, the stabilizing controller will replaces the swinging up controller to balance the pendulum at the vertical upward position.

The paper is organized as follows. System model is provided in section II, including nonlinear dynamic model of the system, linearized model in state-space form and permanent magnet DC motor dynamics. Section III presents controller design. Then, section IV shows simulation and experimental results. Finally, Section 5 concludes this paper.

2. SYSTEM MODELS

2.1 Nonlinear Dynamical Model

The double pendulum is an open-loop, unstable and highly nonlinear system. The objective of the controller is to balance the pendulum at its upward position by the force F acting on the cart. Parameters of the system are showed in table 1

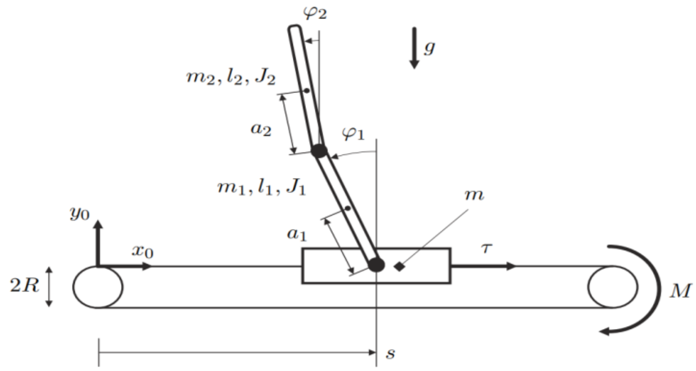

Fig. 1: Reference frames and parameters of pendulum

Figure 1 shows the reference frames and parameters of the system. The movement of the cart is constrained in the *x*-horizontal direction, and the pendulum can rotate in the *x-y* plane. The system has three DOF and can be fully represented using three coordinates: horizontal displacement of the cart, *s*; and rotational displacement of pendulum, φ_1, φ_2. Coordinates of the Centre of Gravity (CoG) of the pendulum is given by:

$$C_1 = \begin{bmatrix} s - a_1 \sin\varphi_1 & a_1 \cos\varphi_1 & 0 \end{bmatrix}^T,$$

$$\dot{C}_1 = [\dot{s} - a_1\dot{\varphi}_1 \cos\varphi_1 \quad -a_1\dot{\varphi}_1 \sin\varphi_1 \quad 0]^T,$$

$$C_2 = \begin{bmatrix} s - l_1 \sin\varphi_1 - a_2 \sin\varphi_2 & l_1 \cos\varphi_1 + a_2 \cos\varphi_2 & 0 \end{bmatrix}^T,$$

$$\dot{C}_2 = \begin{bmatrix} \dot{s} - l_1\dot{\varphi}_1 \cos\varphi_1 - a_2\dot{\varphi}_2 \cos\varphi_2 & -l_1\dot{\varphi}_1 \sin\varphi_1 - a_2\varphi_2 \sin\varphi_2 & 0 \end{bmatrix}^T$$

CoG of the cart is given by:

$$C_{cart} = \begin{bmatrix} s & 0 & 0 \end{bmatrix}^T, \quad \dot{C}_{cart} = \begin{bmatrix} \dot{s} & 0 & 0 \end{bmatrix}^T$$

Table 1: Parameters of the double pendulum

Variable	**Unit**	**Meaning**
φ_1, φ_2	rad	Angular displacement of the pendulum links from the vertical upright position.
s	m	Cart displacement.
J_1, J_2	$kg.m^2$	Moment of inertia of the pendulum links.
m_1, m_2	kg	Mass of the pendulum.
m	kg	The mass of the cart
a_1, a_2	m	The distance from the CoG of the pendulum to the pivot.
g	m / s^2	Acceleration of gravity
d_1	$Nm.s$	Friction coefficient of pendulum
d_2	$Nm.s$	Friction coefficient of pendulum
R_m	Ω	Armature resistance of motor
L_m	H	Armature inductance of motor
K_m	Wb	Emf constant
R	m	Pully radius

Applying Euler-Lagrangian equation to the system yields:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=F-\frac{\partial R}{\partial \dot{q}} \quad (1)$$

where L is the Lagrange function defined as the difference between kinetic and potential energies: L = T – V.

$$\mathbf{D}(q)\ddot{q}+\mathbf{C}\left(q,\dot{q}\right)\dot{q}+\mathbf{G}\left(q\right)=F \quad (2)$$

$$\ddot{q}=\frac{1}{\mathbf{D}(q)}\left[F-\mathbf{C}\left(q,\dot{q}\right)\dot{q}-\mathbf{G}\left(q\right)\right] \quad (3)$$

$$D(q)=\begin{bmatrix} -m_2l_1\cos\varphi_1-m_1a_1\cos\varphi_1 & m_2l_1^2+m_1a_1^2+J_1 & m_2l_1a_2\cos(\varphi_1-\varphi_2) \\ -m_2a_2\cos\varphi_2 & m_2l_1a_2\cos(\varphi_1-\varphi_2) & m_2a_2^2+J_2 \\ m+m_1+m_2 & -m_2l_1\cos\varphi_1-m_1a_1\cos\varphi_1 & -m_2a_2\cos\varphi_2 \end{bmatrix}$$

$$C(q,\dot{q})=\begin{bmatrix} 0 & 0 & m_2l_1a_2\sin(\varphi_1-\varphi_2)\omega_2 \\ 0 & -m_2l_1a_2\sin(\varphi_1-\varphi_2)\omega_1 & 0 \\ 0 & (m_1a_1+m_2l_1)\sin\varphi_1\omega_1 & m_2a_2\sin\varphi_2\omega_2 \end{bmatrix}$$

$$G(q)=\begin{bmatrix} 0 \\ -(m_1a_1+m_2l_1)g\sin\varphi_1 \\ -m_2a_2g\sin\varphi_2 \end{bmatrix}$$

2.2 Linearized Model in State-Space Form

Linearizing the model, the following approximations are applied:

$$\sin(\varphi_1-\varphi_2)=\varphi_1-\varphi_2$$
$$\cos(\varphi_1-\varphi_2)=1$$
$$\varphi_1^2=\varphi_2^2=0$$
$$\cos\varphi_1=\cos\varphi_2=1$$
$$\sin\varphi_1=\varphi_1$$
$$\sin\varphi_2=\varphi_2$$

Defining the state variables as below:

$$x=\begin{bmatrix} s \\ \varphi_1 \\ \varphi_2 \\ \dot{s} \\ \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

Table 2: List of Parameters

Variable	Value	Unit
l_1	0.419	m

l_2	0.484	m
m_1	0.9363	kg
m_2	0.5505	kg
a_1	0.2687	m
a_2	0.2256	m
J_1	0.0243	$kg.m^2$
J_2	0.0183	$kg.m^2$
m	0.2	kg
d_1	0.0003	$Nm.s$
d_2	0.2	$Nm.s$

.

Linearization of (6) around x = 0 and substituting the parameters given in Table 2 into (4), we obtain:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} ; \ x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{bmatrix}^T \ ; y = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T \quad (4)$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -6.9425 & -1.6179 & 0 & 0 & 0 \\ 0 & 54.6574 & -6.2595 & 0 & 0 & 0 \\ 0 & -42.7910 & 37.6742 & 0 & 0 & 0 \end{bmatrix} B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1.1102 \\ -2.9248 \\ 0.30192 \end{bmatrix}$$

2.3 Permanent Magnet DC Motor Dynamics [6]

The relation between the armature current and the armature voltage can be written in Laplace form as:

$$U_m = E_{\text{emf}} + I_m \left(R_m + sL_m\right) \quad (5)$$

where R_m and L_m are resistance and inductance of the rotor, respectively.

The back-emf voltage created by the motor, E_{emf}, is proportional to the rotor speed as:

$$E_{emf} = K_m \dot{\varphi}$$

The electromagnetic torque generated by the DC motor is proportional to the armature current:

$$M_{dt} = K_m I_m$$

We have: $$I_m = \frac{1}{R_m + L_m s}\left(U - E_{emf}\right) = \frac{1/R_m}{T_m s + 1}\left(U - K_m \dot{\varphi}\right) \quad (6)$$

From the above equations, we get the structure diagram of DC motor with feedback current using ACS 712 current sensor as follows:

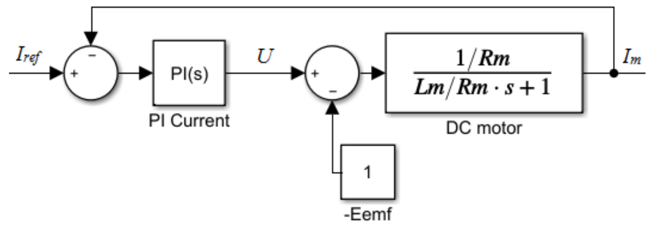

Fig -2: *Closed-Loop DC motor current Control System*

The response rate of the current controller is very fast, so the change from the feedback output is very small. Therefore, the feedback is considered as a noise.

Table 3: List of Parameters.

Parameter	**Value**
DC motor power (P)	120 W
voltage (U)	24 VDC
Current (I)	$5A$
DC motor speed (n)	$1200 Rpm$
rotor inertia (J_m)	$2.10^{-4} Kg.m^2$
pully radius (R)	$0.195\,m$
Armature inductance of motor (L_m)	$0.0281\ H$
Armature resistance of motor (R_m)	$0.34\,\Omega$

. In the classical sense, a PI controller has the following transfer function:

$$W_c = K_p\left(1+K_i\frac{1}{s}\right) = 0.32\left(1+660.16\frac{1}{s}\right) \quad (7)$$

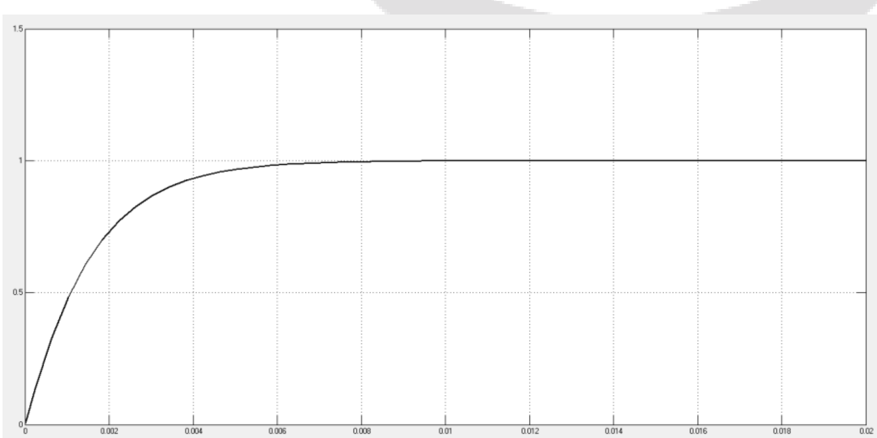

***Fig 3**: Diagram simulating the current controller with the reference set point to 1*

The inner loop needs a fast response. Using PI controller with the above parameters, the system has a Settling Time of 0.008s. Therefore, the designed PI controller meets the requirement.

3. DESIGN CONTROLLERS FOR STABILIZING DOUBLE PENDULUM

3.1 Quadratic optimal regulator problem

The system equation in the state space is represented as

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

We determine the matrix K of the optimal control vector $u = -Kx$ to minimize the performance index:

$$J = \frac{1}{2}\int_0^{+\infty} (x^T Q x + u^T R u)dt \rightarrow \min$$

Where Q and R are weighting matrices. In this problem, we assume that the control vector u(t) is unconstrained. The linear control law given by Eq. (8) is the optimal control law. The matrix K are determined by minimizing the performance index *J*, then u(t) = –Kx(t) is the optimal control signal for any initial state x(0). The block diagram is shown in Fig 4.

In MATLAB, function "lqr" is used to get the corresponding feedback gain matrix K = lqr (A, B, Q, R), where Q is a positive semi-definite real symmetric matrix, R is a positive definite real symmetric matrix. Q and R are selected by experience.
Q = diag ([200, 1, 200, 1, 200, 1]) and R = 1
K=lqr (A, B, Q, R)

Resulting in the optimal gain:

$$K = \begin{bmatrix} 10 & -212.492 & 296.4106 & 14.3510 & -5.7107 & 49.7217 \end{bmatrix}$$

3.2 Swing-Up Control

The model of the double pendulum (3) can be written as a system of second-order ODEs in which the acceleration of the cart $\ddot{s} = u$ serves as input to the system.

$$\ddot{s} = u$$
$$\ddot{\varphi} = \beta(\varphi, \dot{\varphi}, u) \qquad (8)$$

The swing up within a finite time interval t ∈ [0, T] requires to steer the double pendulum from the initial downward equilibrium.

$$s(0) = 0, \ \ \dot{s}(0) = 0, \ \ \varphi(0) = [\pi;\pi]^T, \ \ \dot{\varphi}(0) = 0 \qquad (9)$$

to the terminal upward equilibrium

$$s(T) = 0, \ \ \dot{s}(T) = 0, \ \ \varphi(T) = [0;0]^T, \ \ \dot{\varphi}(T) = 0 \qquad (10)$$

The ODEs (8) together with the boundary conditions (9)–(10) form a two-point boundary value problem (BVP) for the states s, φ and that depends on the input trajectory u. The feedforward control is simply the second time derivative of the desired output trajectory s.

The output trajectory s(t,p) has to satisfy the four boundary conditions (9)–(10), which implies that the output trajectory must be at least once differentiable. The setup function s (t, p) is constructed using the cosine series

$$s(t,p) == -(p_1 + p_3) - (p_2 + p_4)\cos(\frac{\pi t}{T}) + \sum_{i=2}^{5} p_{i-1}\cos(\frac{i\pi t}{nT}) \qquad (11)$$

with the free parameters p = (p1, . . . , p4)

The numerical solution of the BVP (8)–(11) is a standard task in numerics. A particularly convenient way is to use the MATLAB function bvp4c (https://www.mathworks.com/help/matlab/ref/bvp4c.html) designed for the solution of two point BVPs with unknown parameters. Bvp4c returns the trajectories $\varphi(t)=[\varphi_1(t),\varphi_2(t)]^T$ and the parameter set p=(0.0494, 0.206, -0.0824, -0.1558), which yields the output trajectory and the feedforward control (11) $u=\ddot{s}$. Fig.4 shows the nominal trajectories for swing-up times T =2,5s. The maximum acceleration max $\ddot{s}=12.5\ m/s^2$, acceleration max $\dot{s}=2.2\ m/s$, acceleration max s= 0.7m in this case . In contrast to this, the swing-up $T\neq(1.7-2.8\ )s$ time leads to a different swing-up motion and violates the respective constraint. Fig.4. Nominal trajectories for the swing-up of the double pendulum in case T=2.5s when the pendulum approaches a certain area, the stabilizing controller will replaces the swinging up controller to balance the pendulum at the vertical upward position.

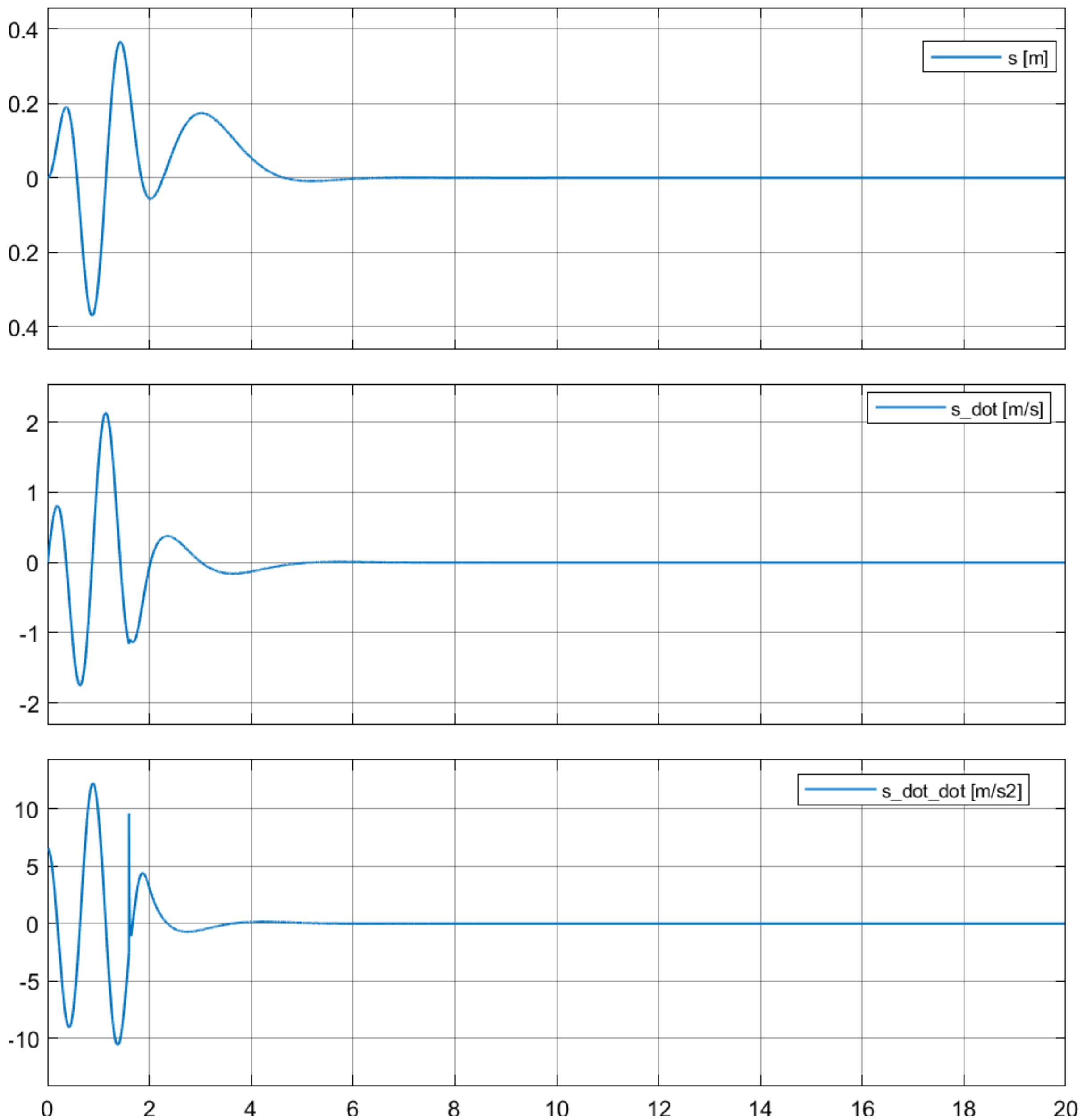

Fig.4. Nominal trajectories for the swing-up of the double pendulum in times T=2.5s

4. SIMULATION RESULTS AND DISCUSSION

4.1 Simulation results

In our research, the model of double pendulum system is pre-designed and simulated on 3D Solidworks software. Then, an experimental setup is built as shown in Fig. 5. The setup consists of a movable cart driven by a DC motor according to the control voltage. The cart can move along a horizontal track. The pendulums are mounted on the cart and can freely rotate around their axis.

Fig. 5: Snapshot of Real plant

Block diagram and simulation result of the controller using swing up in combination with Quadratic optimal control are shown in Figure 6 and Figure 7.

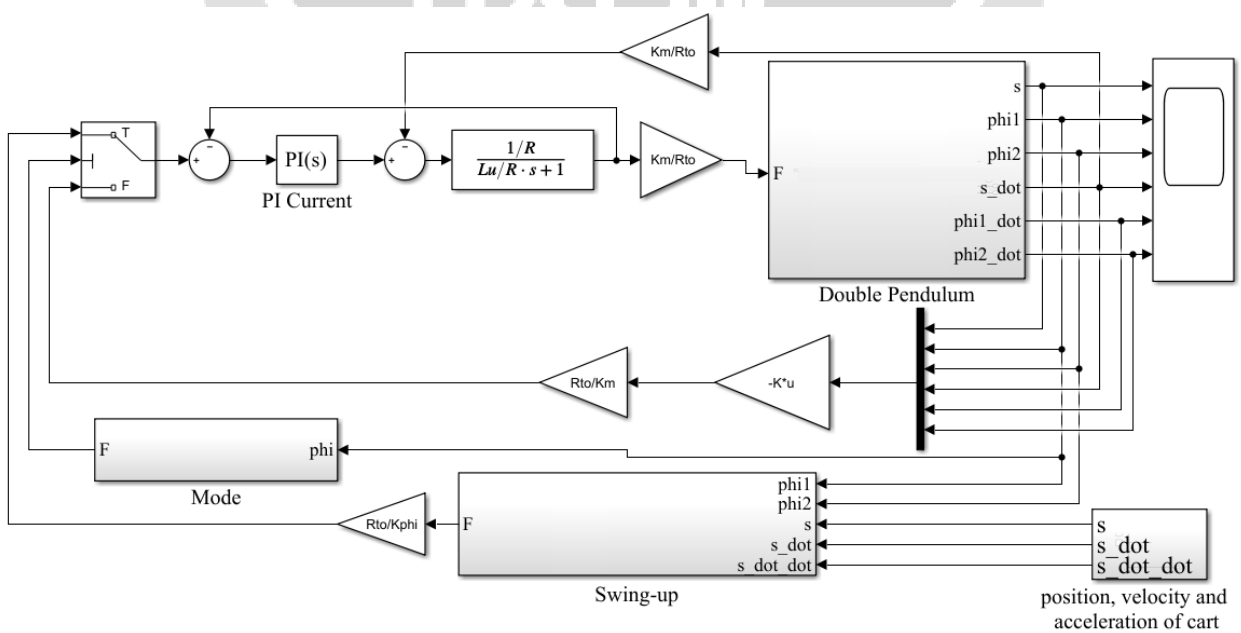

Fig. 6: Block diagram of controllers using Quadratic Optimal Regulator (MATLAB Simulink).

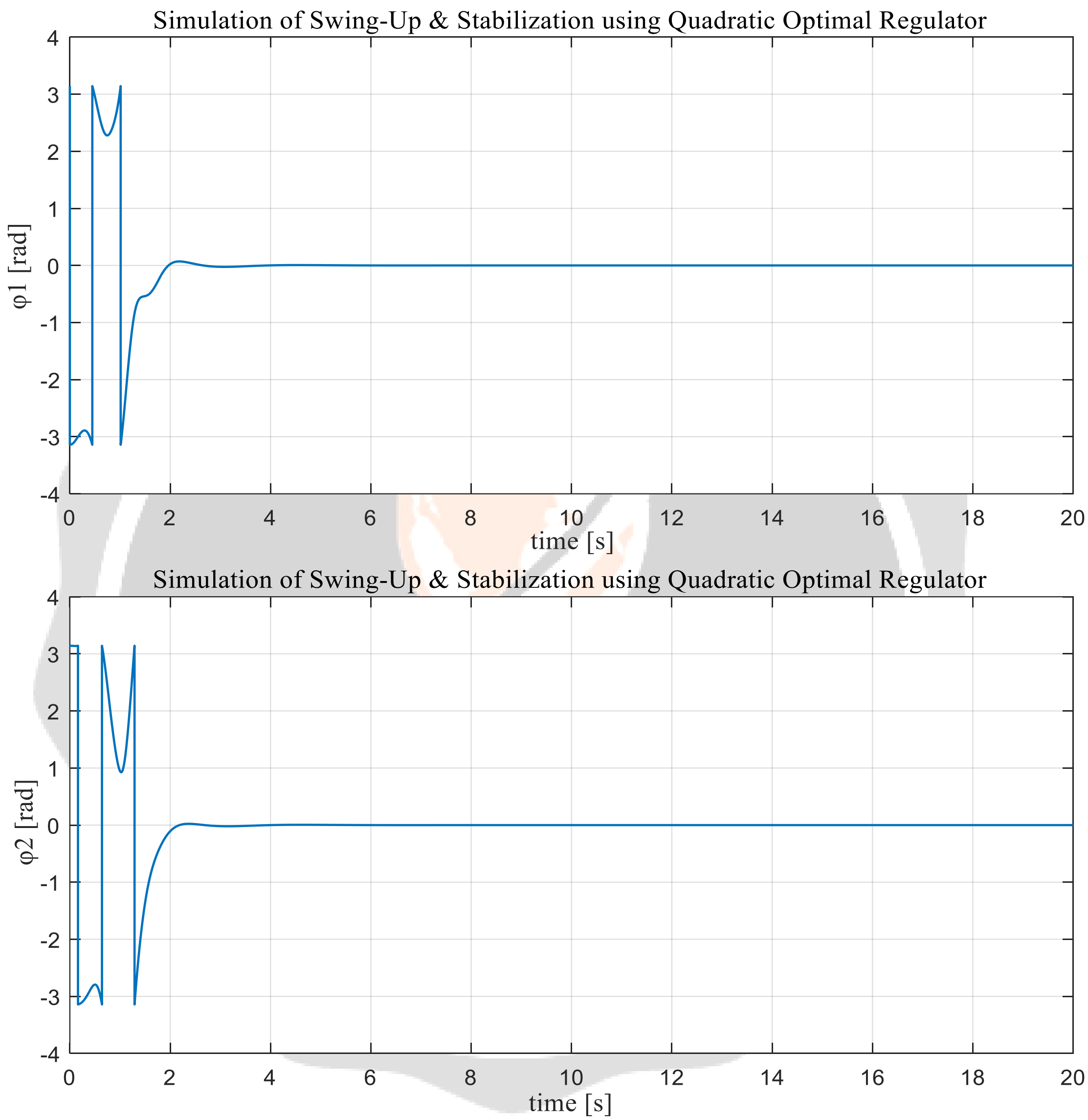

Fig. 7: Simulation of Swing-Up & Stabilization using Quadratic Optimal Regulator

Fig. 6 and Fig. 7 show that the transition time of the system using Quadratic optimal regulator is nearly 3 seconds,.

4.2 Experimental results

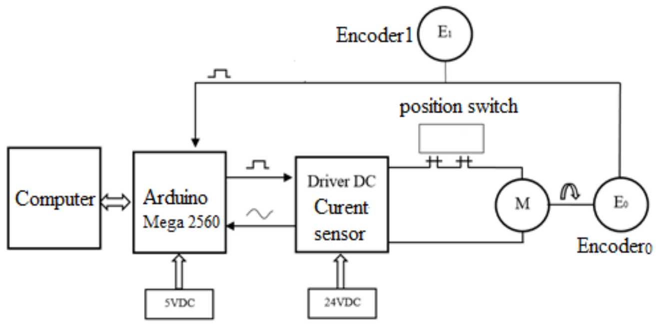

Fig. 8: Block diagram of experimental setup

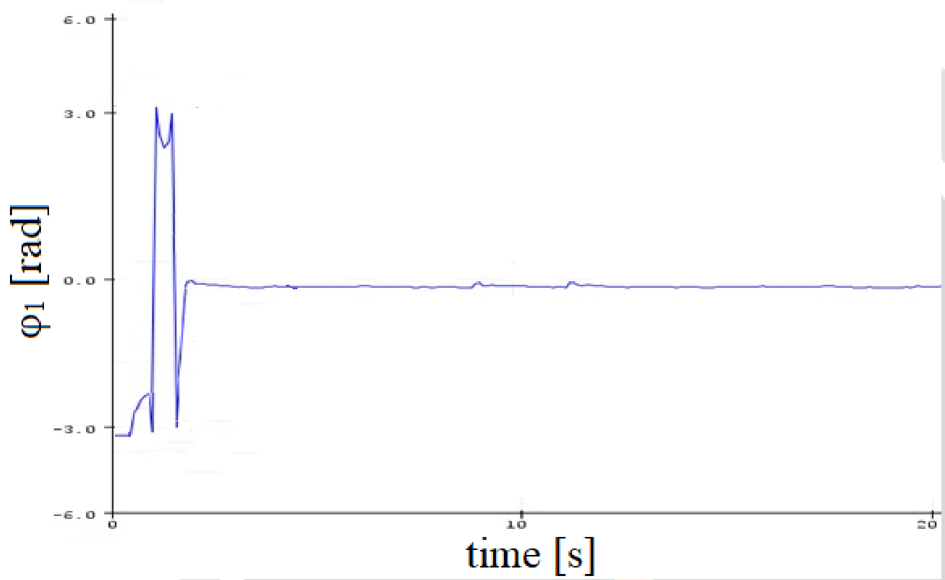

Fig. 9: Experimental Swing-Up & Stabilization using Quadratic Optimal Regulator (φ_1)

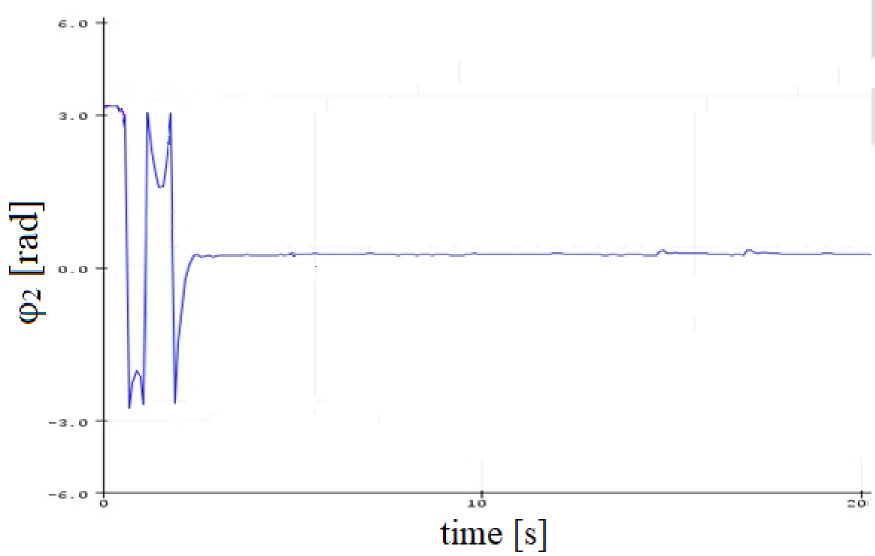

Fig. 10: Experimental Swing-Up & Stabilization (φ_2)

Figure 8 shows the block diagram of the experimental setup. Experimental results of the controller using swing up combined with Quadratic optimal control *(φ_1, φ_2)* in Figures 9, 10. It can be seen that feedforward control input $u = \ddot{s} = s\_dot\_dot$ (in Figure 6) and the stabilizing controller is able to move and balance the pendulums from their stable equilibrium point, $x=[0, \pi, \pi,0,0,0]^T$, to their unstable equilibrium point, $x=[0,0,0,0,0,0]^T$.

5. CONCLUSIONS

The proposed controller has achieved that the system is able not only to swing up and balance the pendulum from downward position to the upward equilibrium point, but also to return the cart to its original position on the rail. The pendulum is stable at its upward position. This proves that the control algorithm is effective. Simulation and experimental results are almost similar. In experimental results, however, the pendulum still oscillates slightly around the equilibrium position. This could be due to the dynamic uncertainty, pinion backlash, motor dead-zone, magnetic hysteresis, and other mechanical imperfections.

Our future research is control design for the triple link pendulum system.

6. ACKNOWLEDGEMENT

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6. REFERENCES

[1].ASCHER, U., MATTHEIJ, R., AND RUSSELL, R. 1995. Numerical Solution of Boundary Value Problems for Ordinary Differential Equations. SIAM, Philadelphia, PA, USA.

[2].KIERZENKA, J. AND SHAMPINE, L. 2001. A BVP solver based on residual control and the MATLAB PSE. ACM Trans. Math. Soft. 27, 299–316.

[3].E. A. Baran, E. Golubovic, A. Sabanovic, "A New Functional Observer to Estimate Velocity, Acceleration and Disturbance for Motion Control Systems", International Symposium on Industrial Electronics (ISIE), pp. 384-389, July 2010.

[4].K. Ohnishi, M. Shibata, T. Murakami, "Motion Control for Advanced Mechatronics", IEEE/ASME Transactions on Mechatronics (APEC), Vol. 1, No. 1, pp.56-67, March 1996.

[5].A. Sabanovic, K. Ohnishi, "Motion Control Systems", Willey Publisher (2011).

[6].Tran Thien Dung, Nguyen Nam Trung, Nguyen Van Lanh, “Control design using backstepping technique for a cart-double pendulum system” International Journal of Engineering and Applied Sciences (IJEAS) ISSN: 2394-3661, Volume-6, Issue-1, January 2019.

[7].Nguyen Trong Toan, Nguyen Nam Trung, “Auto-tuning controllers of a class of plants using gradient descent algorithm” International Journal of Engineering and Applied Sciences (IJEAS) ISSN: 2394-3661, Volume-6, Issue-6, June 2019.

[8].Manjeet And Pooja Khatri trajectory control of two link robotic manipulator using PID, Volume-3, Issue-5, Nov-2013.

[9].Nguyen Nam Trung, On the feedback-feedforward control for robot manipulator used disturbance estimator. *IJARIIE-ISSN(O)-2395-4396*, Vol-9 Issue-3 2023.

[10]. Ngo Phuong Thanh, Nguyen Nam Trung, "The Fan and Plate Angular Position Control Techniques and Applications," *SSRG International Journal of Electrical and Electronics Engineering*, vol. 8, no. 6, pp. 38-42, 2021.

[11]. Mai Trung Thai, Nguyen Trong Toan, Nguyen Nam Trung., Design The Temperature Controller Application For Heat Transfer System In The Laboratory, SSRG International Journal of Electrical and Electronics Engineering 8(6) (2021) 23-27.

[12]. K.J. Åström, K. Furuta, 2000, “Swinging up a pendulum by energy control”, Automatica 36, no.2, pp 287-295.

[13]. Chen, X., Zhou, H., Ma, R & et al. “Linear Motor Driven Inverted Pendulum and LQR Controller Design”, Proceedings of the IEEE International Conference on Automation and Logistics, 2007, pp. 1750-1754.

[14]. N. Dhang and S. Majumdar, “Optimal Control of Nonlinear Inverted Pendulum System, “Using PID Controller and LQR: Performance Analysis Without and With Disturbance Input”, International Journal of Automation and Computing 11(6), December 2014.

[15]. M. H. Arbo, P. A. Raijmakers, and V. M. Mladenov, “Applications of neural networks for control of a double inverted pendulum,” in 12th Symposium on Neural Network Applications in Electrical Engineering (NEUREL), Nov 2014, pp. 89–92.
[16]. Tomohiro Henmi, Mingcong Deng, Akira Inoue **,** Nobuyuki Ueki, and Yoichi Hirashima,” Swing-up Control of a Serial Double Inverted Pendulum” Proceeding of the 2004 American Control Conference Boston, Massachusetts June 30 - July 2, 2004.
[17]. Alexander Bogdanov**, ”**Optimal Control of a Double Inverted Pendulum on a Cart**”,** Technical Report CSE-04-006 December 2004.\
[18]. K. Graichen a & M. Zeitz (2008), "Feedforward control design for finite-time transition problems of non-linear MIMO systems under input constraints, "International Journal of Control, 81:3, 417-427.