DYNAMICS AND BEHAVIOR OF HIGHER ORDER AND NONLINEAR RATIONAL DIFFERENCE EQUATION

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ABSTRACT

In the article we discuss about the periodicity solution of the difference equation and also the global stability of the positive solutions of nonlinear difference equation

$$y_{n+1} = Py_n + Qy_{n-k} + Ry_{n-l} + \frac{by_{n-k}}{dy_{n-k} - ey_{n-l}}, n = 0,1,2,\dots$$

Where the co-efficients $, Q, R, b, d, e \in (0,\infty)$ *, while k and l are positive integers. The initial conditions* $y_{-l},\dots,y_{-k},\dots,y_{-1},y_0$ *are arbitrary positive real numbers such that* $k < l$*. Numerical examples will be given to illustrate our results. Here we used boundedness positive integers and also the parameter which is used for theorems and lemmas.*

Keyword

Difference equations, prime period two solution, boundedness character, locally asymptotically stable, global stability.

1.INTRODUCTION

The qualitative study of difference equations is a fertile research area and increasingly attracts many mathematicians. This topic draws its importance from the fact that many real life phenomena are modelled using difference equations. Examples from economy, biology, etc. can be found in [1-6]. It is known that nonlinear difference equations are capable of producing a complicated behaviour regardless its order. This can be easily seen from the family $y_{n+1} = g_\mu(y_n), \mu > 0, n \geq 0$. This behaviour is ranging according to the value of μ, from the existence of a bounded number of periodic solutions to chaos.

There has been a great interest in studying the global attractivity, the boundedness character and the periodicity nature of nonlinear difference equations. For example, in the articles [7-13] closely related global convergence results were obtained which can be applied to nonlinear difference equations in proving that every solution of these equations converges to a period two solution. For other closely related results (see [14-23]) and the reference cited there in. The study of these equations is challenging and rewarding and still in its infancy. We believe that the nonlinear rational difference equations are of paramount importance in their own right.

Furthermore the results about such equations offer prototypes for the development of the basic theory of the global behaviour of nonlinear difference equations.

The objective of this article is to investigate some qualitative behaviour of the solutions of the nonlinear difference equation

$$y_{n+1} = Py_n + Qy_{n-k} + Ry_{n-l} + \frac{by_{n-k}}{dy_{n-k} - ey_{n-l}}, n = 0,1,2, \qquad (1)$$

Where the coefficients $P, Q, R, b, d, e \in (0, \infty)$,while k and l are positive integers. The initial conditions$y_{-l}, \dots, y_{-k}, \dots \dots y_{-1}, y_0$ are artrary positive real numbers such that $k < l$. Note that the special cases of Eq.(1) have been studied discussed in [24] when $Q = R = 0$an $k = 0, l = 1, b$,is replaced by $-b$and in [25] when $Q = R = 0$and , $b$is replaced by $-b$ and in [26] when $P = R = 0, b$ is replaced by$-b$ and in [27] when $Q = R = 0, l = 0$

Our interest now is to study behaviour of the solutions of Eq.(1) in its general form. For the related work (see [28-30]). Let us now recall some well known results [2] which will be useful in the sequel.

Our goal in this paper is to investigate the global stability character and the periodicity of solutions of the recursive sequence

$$y_{n+1} = a + \frac{b\,y_{n-l} + cy_{n-k}}{d\,y_{n-l} + ey_{n-k}} \qquad (I)$$

Where the parameters a, b, c, d and e the positive real numbers and the initial conditions $y_{-t}, y_{-t+1}, \dots, y_{-1}$ and y_0 are positive real numbers where $t = \max\{l, k\}$ and $l \neq k$.

1.1 Definition

Consider a difference equation in the form $y_{n+1} = F(y_n, y_{n-k}, y_{n-l}),\ \ n = 0,1,2,$ (2)
Where $F$is a continuous function, while k and l are positive integers such that $k < l$. An equilibrium point $\tilde{y}$ of this equation is a point that satisfies the condition $\tilde{y} = F(\tilde{y}, \tilde{y}, \tilde{y})$. That is, the constant sequence $\{y_n\}$ with $y_n = y$for all $n \geq -k \geq -l$ is a solution of that equation

1.2 Definition

let $\tilde{y} \in (0, \infty)$ be an equilibrium point of Eq. (2). Then we have

(i) An equilibrium point $\tilde{y}$ of Eq. (2) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $y_{-l}, \dots, y_{-k}, \dots \dots, y_{-1}, y_0 \in (0, \infty)$with $|y_{-l} - \tilde{y}| + \cdots + |y_{-k} - \tilde{y}| + \cdots + |y_{-1} - \tilde{y}| + |y_0 - \tilde{y}| < \delta$, then $|y_n - \tilde{y}| < \varepsilon$ for all $n \geq -k \geq -l$.

(ii) An equilibrium point $\tilde{y}$ of Eq. (2) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that, if $y_{-l}, \dots, y_{-k}, \dots, y_{-1}, y_0 \in (0, \infty)$with $|y_{-l} - \tilde{y}| + \cdots + |y_{-k} - \tilde{y}| + \cdots + |y_{-1} - \tilde{y}| + |y_0 - \tilde{y}| < \gamma$, then $\lim_{n\to\infty} x_n = \tilde{y}$.

(iii) An equilibrium point $\tilde{y}$ of Eq. (2) is called a global attractor if for every $y_{-l}, \dots, y_{-k}, \dots \dots, y_{-1}, y_0 \in (0, \infty)$we have $\lim_{n\to\infty} x_n = \tilde{y}$.

(iv) An equilibrium point $\tilde{y}$ of Eq. (2) is called globally asymptotically stable if it is locally stable and a global attractor.

(v) An equilibrium point $\tilde{y}$ of Eq. (2) is called unstable if it is not locally stable.

1.3 Definition

A sequence $\{y_n\}_{n=-l}^{\infty}$ is said to be periodic with period r if $y_{n+r} = y_n$ for all $n \geq -p$. A sequence $\{y_n\}_{n=-l}^{\infty}$ is said to be periodic with prime period r if $r$is the smallest positive integer having this property.

1.4 Definition

Eq. (2) is called permanent and bounded if there exists numbers m and M with $0 < m < M < \infty$such that for any initial conditions $y_{-l}, \dots, y_{-k}, \dots \dots, y_{-1}, y_0 \in (0, \infty)$ there exists a positive integer Nwhich depends on these initial conditions such that $m \leq y_n \leq M$for all $n \leq N$.

1.5 Definition

The linearized equation of Eq. (2) about the equilibrium point y is defined by the equation

$$z_{n+1} = \rho_0 z_n + \rho_1 z_{n-k} + \rho_2 z_{n-1} = 0, \tag{3}$$

Where

$$\rho_0 = \frac{\partial F(\tilde{y},\tilde{y},\tilde{y})}{\partial y_n}, \qquad \rho_1 = \frac{\partial F(\tilde{y},\tilde{y},\tilde{y})}{\partial y_{n-k}}, \qquad \rho_2 = \frac{\partial F(\tilde{y},\tilde{y},\tilde{y})}{\partial y_{n-l}}$$

The characteristic equation associated with Eq. (3) is

$$\rho(\lambda) = \lambda^{l+1} - \lambda^{l} - \lambda^{l-k} - \rho_2 = 0. \tag{4}$$

2.0 The local stability of the solutions

The equilibrium point $\tilde{y}$ of Eq. (1) is the positive solution of the equation

$$\tilde{y} = (P+Q+R)\tilde{y} + \frac{b\tilde{y}}{(d-e)\tilde{y}} \tag{5}$$

Where $d \neq e$. If $[(P+Q+R) - 1.(e-d)] > 0$, then the only positive equilibrium point $\tilde{y}$ of Eq. (1) is given by

$$\tilde{y} = \frac{b}{[(P+Q+R)-1.(e-d)} \tag{6}$$

Let us now introduce a continuous function

$F:(0,\infty)^3 \to (0,\infty)$ which is defined by

$$F(u_0,u_1,u_2) = Pu_0 + Qu_1 + Ru_2 + \frac{bu_1}{(du_1 - eu_2)}, \tag{7}$$

Provided $du_1 \neq eu_2$ consequently, we get

$$\begin{cases} \frac{\partial F(\tilde{y},\tilde{y},\tilde{y})}{\partial u_0} = P = \rho_0\,, \\ \frac{\partial F(\tilde{y},\tilde{y},\tilde{y})}{\partial u_1} = Q - \frac{e[(P+Q+R)-1]}{(e-d)} = \rho_1\,, \\ \frac{\partial F(\tilde{y},\tilde{y},\tilde{y})}{\partial u_2} = R - \frac{e[(P+Q+R)-1]}{(e-d)} = \rho_2\,, \end{cases} \tag{8}$$

Where $e \neq d$. Thus the linearized equation of Eq. (1) about $\tilde{y}$ takes the for

$Z_{n+1} - \rho_0 Z_n - \rho_1 Z_{n-k} - \rho_1 Z_{n-l} = 0$, Where ρ_0, ρ_1 and ρ_2 are given by (8).

2.1 Local stability of the equilibrium point of (I)

This section deals with study of the local stability character of the equilibrium point of $y_{n+1} = a + \frac{by_{n-l}+cy_{n-k}}{dy_{n-l}+ey_{n-k}}$

Theorem 2.1

Assume that $2|be - dc| < (d+e)(a(d+e)+b+c)$. Then the positive equilibrium point of $y_{n+1} = a + \frac{by_{n-l}+cy_{n-k}}{dy_{n-l}+ey_{n-k}}$ is locally asymptotically stable.

Proof

The only positive equilibrium point of $y_{n+1} = a + \frac{by_{n-l}+cy_{n-k}}{dy_{n-l}+ey_{n-k}}$ is given by $\bar{y} = a + \frac{b+c}{d+e}$

Let $f:(0,\infty)^2 \to (0,\infty)$ be a continuous function defined by

$$f(u,v) = a + \frac{bu+cv}{du+ev} \qquad \text{(i)}$$

We have

$\frac{\partial f(u,v)}{\partial u} = \frac{(be-dc)v}{(du+ev)^2}$ and $\frac{\partial f(u,v)}{\partial v} = \frac{(dc-be)u}{(du+ev)^2}$

Then we see that

$$\frac{\partial f(\bar{y},\bar{y})}{\partial u} = \frac{(be-dc)}{(d+e)^2\bar{y}} = \frac{(be-dc)}{(d+e)(a(d+e)+b+c)} = -a_1$$

$$\frac{\partial f(\bar{y},\bar{y})}{\partial v} = \frac{(dc-be)}{(d+e)(a(d+e)+b+c)} = -a_0$$

Then the linearized equation of (2) about $\bar{y}$ is

$$x_{n+1} + a_1 x_{n-1} + a_0 x_{n-k} = 0, \qquad \text{(ii)}$$

Whose characteristic equation is

$$\alpha^{k+1} + a_1\alpha^{k-1} + a_0 = 0 \qquad \text{(iii)}$$

The Eq. (ii) is asymptotically stable if all the roots of (iii) lie in the open disc $|\alpha| < 1$, that is if

$|a_1| + |a_0| < 1$,

$$\left|\frac{(be-dc)}{(d+e)(a(d+e)+b+c)}\right| + \left|\frac{(dc-be)}{(d+e)(a(d+e)+b+c)}\right| < 1,$$

And so $\qquad 2\left|\frac{(be-dc)}{(d+e)(a(d+e)+b+c)}\right| < 1,$

Or $\qquad 2|be-dc| < (d+e)(a(d+e)+b+c).$

The proof is complete.

3.0 Periodic solutions

In this section, we discuss about the existence of periodic solutions of

$$y_{n+1} = Py_n + Qy_{n-k} + Ry_{n-l} + \frac{by_{n-k}}{dy_{n-k} - ey_{n-l}}, n = 0,1,2, \dots$$

3.1 Theorem

If k and l are both even positive integers, then $y_{n+1} = Py_n + Qy_{n-k} + Ry_{n-l} + \frac{by_{n-k}}{dy_{n-k}-ey_{n-l}}$, $n = 0,1,2 \dots$ has no prime period two solution.

Proof

Assume that there exist distinct positive solutions

$$\dots, A, B, A, B, \dots \qquad (9)$$

of prime period two of $\quad y_{n+1} = Py_n + Qy_{n-k} + Ry_{n-l} + \frac{by_{n-k}}{dy_{n-k}-ey_{n-l}}, \qquad n = 0,1,2, \dots$

If k and l are both even positive integers, then $y_n = y_{n-k} = y_{n-l}$. It follows from

$$y_{n+1} = Py_n + Qy_{n-k} + Ry_{n-l} + \frac{by_{n-k}}{dy_{n-k} - ey_{n-l}}, \qquad n = 0,1,2, \ldots \text{that}$$

$$A = (P + Q + R)B - \frac{b}{(e-d)} \tag{10}$$

and

$$B = (P + Q + R)A - \frac{b}{(e-d)}. \tag{11}$$

By subtracting (11) from (10), we get

$$(P + Q + R)A - (P + Q + R)B - \frac{b}{(e-d)} + \frac{b}{(e-d)} = (B - A)$$

$$(P + Q + R)A(P + Q + R)B = (B - A)$$

$$(A - B)[(P + Q + R)] = (B - A)$$

$$(A - B)[(P + Q + R)] - (B - A) = 0$$

$$(A - B)[(P + Q + R)] + (A - B) = 0$$

$$(A - B)[(P + Q + R) + 1] = 0 \tag{12}$$

Since $P + Q + R + 1 \neq 0$, then $A = B$.

This is a contradiction. Thus, the proof is now completed.

3.2 Theorem

If $k$and $l$are both odd positive integers and $P + 1 \neq Q - R$, then

S$y_{n+1} = Py_n + Qy_{n-k} + Ry_{n-l} + \frac{by_{n-k}}{dy_{n-k} - ey_{n-l}}$, $n = 0,1,2, \ldots$has no prime period two solution.

Proof

Following the proof of theorem 1, we deduce that if *k* and *l* are both odd positive integers, then $y_{n+1} = y_{n-k} = y_{n-l}$, it follows from $y_{n+1} = Py_n + Qy_{n-k} + Ry_{n-l} + \frac{by_{n-k}}{dy_{n-k} - ey_{n-l}}, n = 0,1,2, \ldots.$ that

$$A = PB + (Q + R)A + \frac{b}{(e-d)}, \tag{13}$$

and

$$B = PA + (Q + R)B + \frac{b}{(e-d)}, \tag{14}$$

By subtracting (14) from (13), we get

$$PA - PB + (Q + R)B - (Q + R)A + \frac{b}{(e-d)} - \frac{b}{(e-d)} = (B - A)$$

$$PA - PB + (Q + R)B - (Q + R)A = (B - A)\backslash$$

$$A[P - (Q + R)] - B[P - (Q + R)] = (B - A)$$

$(A-B)[P-(Q+R)]-(B-A)=0$

$(A-B)[P-(Q+R)]+(A-B)=0$

$$(A-B)[(P-(Q+R))+1]=0 \quad (15)$$

Since $-(Q+R)+1\neq 0$,then $A=B$

This is a contradiction. Thus, the proof is now completed.

3.3 Theorem

If $k$is even and $l$is odd positive integers, then $y_{n+1}=Py_n+Qy_{n-k}+Ry_{n-l}+\frac{by_{n-k}}{dy_{n-k}-ey_{n-l}}, n=0,1,2,\ldots$ Shas prime period two solution if the condition

$$(1-R)(3e-d)(1-R)(3e-d)<(e+d)(A+B), \quad (16)$$

is valid, provided $C<1$and$e(1-C)-d(A+B)>0.$

Proof

If $k$is even and $l$is odd positive integers, then $y_n=y_{n-k}$and $y_{n+1}=y_{n-l}$it follows from

$y_{n+1}=Py_n+Qy_{n-k}+Ry_{n-l}+\frac{by_{n-k}}{dy_{n-k}-ey_{n-l}}, n=0,1,2\ldots$ ....that

$$A=(P+Q)B+RA-\frac{bB}{(eA-dB)}, \text{ and } B=(P+Q)A+RB-\frac{bA}{(eB-dA)} \quad (17)\&\ (18)$$

$$eA^2-dAB=e(P+Q)AB-d(P+Q)B^2+eRA^2-RdAB-bB, \quad (19)$$

$$eB^2-dAB=e(P+Q)AB-d(P+Q)A^2+eRB^2-RdAB-bA, \quad (20)$$

By subtracting (20) from (22), we get

$$A+B=\frac{b}{[e(1-R)-d(P+Q)]}, \quad (21)$$

Where$e(1-R)-d(P+Q)e>0.$ By adding (19) and (20), we obtain

$$AB=\frac{eb^2(1-R)}{(e+d)[(1-R)+(P+Q)][e(1-R)-d(P+Q)^2} \quad (22)$$

Where $R<1$. Assume that P and Q are two positive distinct real roots of the quadratic equation.

$$t^2-(A+B)t+AB=0(23)$$

Thus, we deduce that $(A+B)^2>4AB$

Substituting (21) and (22) into (14). Thus, the proof is now completed.

3.4 Theorem

If k is odd and l is even positive integers, then

$y_{n+1} = Py_n + Qy_{n-k} + Ry_{n-l} + \frac{by_{n-k}}{dy_{n-k} - ey_{n-l}}, n = 0,1,2 \ldots \ldots$

has prime period two solution if the condition

$$(P + R)(3e - d) < (e + d)(1 - Q), \tag{24}$$

Is valid, provided $Q - 1$ and $d(1 - Q) - e(P + R) > 0$.

Proof

If k is odd and l is even positive integers, then $y_{n+1} = y_{n-l}$ and $y_n = y_{n-l}$

If follows $y_{n+1} = Py_n + Qy_{n-k} + Ry_{n-l} + \frac{by_{n-k}}{dy_{n-k} - ey_{n-l}}, n = 0,1,2 \ldots \ldots$

That $A = (P + Q)B + QA - \frac{bA}{(eB - dA)}$ (25)

And $B = (P + Q)A + QB - \frac{bB}{(eA - dB)}$ (26)

Consequently, we get

$$A + B = \frac{b}{[d(1-Q) - e(p+R)]} \tag{27}$$

Where $d(1 - Q) - e(P + R) > 0$,

$$AB = \frac{eb^2(P+R)}{(e+d)[(1-Q)+(P+R)[d(1-R)-e(P+R)]^2} \tag{28}$$

Where $Q < 1$.substituting (27) and (28) into (23), we get the condition (24) Thus, the proof is now completed.

4.0 Boundedness of solutions of (I)

In this section, we investigate the boundedness of the positive solutions of

$y_{n+1} = Py_n + Qy_{n-k} + Ry_{n-l} + \frac{b\,y_{n-k}}{dy_{n-k} - ey_{n-l}}$, *n=0, 1, 2, ...*

and boundeness nature of the solutions of (I),

Theorem 4.1

Every solution of $y_{n+1} = a + \frac{by_{n-l} + cy_{n-k}}{dy_{n-l} + ey_{n-k}}$ is bounded and persists.

Proof

Let $\{x_n\}_{n=-t}^{\infty}$ be a solution of $y_{n+1} = a + \frac{by_{n-l} + cy_{n-k}}{dy_{n-l} + ey_{n-k}}$. it follows from $y_{n+1} = a + \frac{by_{n-l} + cy_{n-k}}{dy_{n-l} + ey_{n-k}}$ that

$$y_{n+1} = a + \frac{by_{n-l} + cy_{n-k}}{dy_{n-l} + ey_{n-k}} \le a + \frac{\max\{b, c\}}{\min\{d, e\}},$$

Then $y_{n+1} \le a + \frac{\max\{b,c\}}{\min\{d,e\}} = M$ for all $n \ge 1$. (29)

Also, we see from (1.1) that

$$y_{n+1} = a + \frac{by_{n-l} + cy_{n-k}}{dy_{n-l} + ey_{n-k}} \ge a + \frac{\min\{b,c\}}{\max\{d,e\}},$$

Then $y_{n+1} \geq a + \frac{\min\{b,c\}}{\max\{d,e\}} = m$ for all $n \geq 1$ (30)

Thus we get from (29) and (30) that

$$a + \frac{\min\{b,c\}}{\max\{d,e\}} \leq y_{n+1} \leq a + \frac{\max\{b,c\}}{\min\{d,e\}} \quad \text{for all} \quad n \geq 1$$

Thus, the solution is bounded and persists.

Theorem4.2

Let $\{y_n\}$ be a solution of equ (1). Then the following statements are true:

(i) Suppose $b < d$ and for some $N \leq 0$, the initial conditions $y_{N-l+1}, \dots, y_{N-k+1}, \dots, y_{N-1}, y_N \in \left[\frac{b}{d}, 1\right]$ are valid ,then for $b \neq e$ and $d^2 \neq be$, we have the inequality

$$\frac{b}{d}(P+Q+R) + \frac{b^2}{(d^2-be)} \leq y_n \leq (P+Q+R) + \frac{b}{(b-e)}, \tag{31}$$

For all $n \geq N$.

(ii) Suppose $b > d$ and for some $N \geq 0$, the initial conditions $y_{N-l+1}, \dots, y_{N-k+1}, \dots, y_{N-1}, y_N \in \left[1, \frac{b}{d}\right]$ are valid ,then for $b \neq e$ and $d^2 \neq be$, we have the inequality

$$(P+Q+R) + \frac{b}{(b-e)} \leq y_n \leq \frac{b}{d}(P+Q+R) + \frac{b^2}{(d^2-be)} \tag{32}$$

For all $n \geq N$.

Proof

First of all, if for some $N \geq 0, \frac{b}{d} \leq y_N \leq 1$and $b \neq e$, we have

$$y_{N+1} = Py_N + Qy_{N-k} + Ry_{N-l} + \frac{by_{N-k}}{dy_{N-k} - ey_{N-l}} \leq P + Q + R + \frac{by_{N-k}}{dy_{N-k} - ey_{N-l}} \tag{33}$$

But, it is easy to see that $dy_{N-k} - ey_{N-l} \geq b - e$, then for $b \neq e$, we getS

$$y_{N+1} \leq (P+Q+R) + \frac{b}{(b-e)}. \tag{34}$$

Similarly, we can show that

$$y_{N+1} \geq (P+Q+R) + \frac{by_{N-k}}{dy_{N-k} - ey_{N-l}} \tag{35}$$

But, one can see that $dy_{N-k} - ey_{N-l} \leq \frac{d^2-be}{d}$, then for $d^2 \neq be$, we get

$$y_{N+1} \geq \frac{b}{d}(P+Q+R) + \frac{b^2}{(d^2-be)} \tag{36}$$

From (34) and (36) we deduce for all $n \leq N$ that the inequality (31) is valid. Hence, the proof of part (i) is completed.

Similarly, if $1 \leq y_n \leq \frac{b}{d}$, we can prove part (ii) which is omitted here for convenience. Thus, the proof is now completed.

Hence the proof

5 Conclusions

We have discussed some properties of the nonlinear rational difference eq.(1), namely the periodicity,s the boundedness and global stability of the positive solutions of this equation. We gave some figures to illustrate the behaviour of these solutions. Our results in this article can be considered as a more generalization. Than the results obtained in refs [5].

6 References

[1] W. T. Li, H.R. Sun, Dynamics of a rational difference equation, Appl. Math. Comput . 163(2005) 577-591.

[2] M. Saleh, S. Abu-Bala,Dynamics of a higher order rational difference equation, Appl. Math. Comput. 181(2006) 84-102.

[3] M.A. El-Moneam, On the dynamics of the higher order nonlinear rational difference equation, Math. Sci. Lett. 3(2)(2014) 121-129

[4] M.A. El-Moneam, On the dynamics of the solutions of the rational recursive sequences, Br. J. Math. Comput. Sci. 5(5)()2015) 654-665.

[5] M.A. El-Moneam , E.M.E. Zayed, Dynamics of the rational difference equation $y_{n+1} = Py_n + Qy_{n-k} + Ry_{n-l} + \frac{by_{n-k}}{dy_{n-k} - ey_{n-l}}$, DCDIS Ser. A:Math.Anal. 21(2014) 317-331.