FIXED POINT THEOREMS FOR INTUITIONISTIC FUZZY METRIC SPACE

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ABSTRACT

The intuitionist fuzzy fixed point theory has become an area of interest for specialists in fixed point theory as intuitionistic fuzzy mathematics has covered new possibilities for fixed point theorists. In this paper, we give some conditions of which four self mappings for Intuitionistic fuzzy metric space have a unique common fixed point.

Keywor: - *Fixed Points, Fuzzy Sets, Fuzzy Metric Spaces, Intuitionistic Fuzzy Sets and Intuitionistic Fuzzy Metric Spaces.*

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1. Introduction

The Fuzzy Mathematics commenced with the introduction of the notion of fuzzy sets by Zadeh L. [15] as a new way to represent the vagueness in everyday life. In mathematical programming, problems are expressed as optimizing some goal function given certain constraints and there are real life problems that consider multiple objectives. Generally, it is very difficult to get a feasible solution that brings us to the optimum of all objective functions. A possible method of resolution, that is quite useful, is the one using fuzzy sets by Turkoglu D. and Rhoades B.E. [14]. Atanassov [6] introduced the notion of intuitionistic fuzzy sets by generalizing the notion of fuzzy set by treating membership as a fuzzy logical value rather than a single truth value. All result which holds of fuzzy sets can be transformed intuitionistic fuzzy sets but converse need not be true. Intuitionistic fuzzy set can be viewed in the context as a proper tool for representing hesitancy concerning both membership and non−membership of an element to a set. Fixed point and common fixed point properties for mappings defined on fuzzy metric spaces, intuitionistic fuzzy metric spaces have been studied by many authors like []. Most of the properties which provide the existence of fixed points and common fixed points are of linear contractive type conditions. Saadati and park [12] studied the concept of intuitionistic fuzzy metric space and its applications. Further, they introduced the notion of Cauchy sequences in an intuitionistic fuzzy metric space and proved the well-known fixed point theorem of Banach and Edelstein extended to intuitionistic fuzzy metric space with the help of Grabiec [57] gave a generalization of Jungck's common fixed point theorem Jungck [66] to intuitionistic fuzzy metric spaces. They first formulated the definition of weakly commuting and R-weakly commuting mappings in intuitionistic fuzzy metric spaces and proved the intuitionistic fuzzy version of Pant's theorem. In this paper, we give some conditions of which four self mappings for Intuitionistic fuzzy metric space have a unique common fixed point.

2. Preliminaries

Definition 2.1: A binary operation $*: [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous $t-$norm if $*$ is satisfying the following conditions

(a) $*$ is commutative and associative;
(b) $*$ is continuous;
(c) $a * 1 = a$ for all $a \in [0,1]$;
(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$.

Definition 2.2: A binary operation $\lozenge: [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t −conorm if $\lozenge$ is satisfying the following conditions

(a) $\lozenge$ is commutative and associative;
(b) $\lozenge$ is continuous;
(c) $a \lozenge 0 = a$ for all $a \in [0,1]$;
(d) $a \lozenge b \leq c \lozenge d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$.

Definition 2.3: A 5 −tuple $(X, \mathcal{M}, N, *, \lozenge)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t − norm, $\lozenge$ is a continuous t −conorm and $\mathcal{M}, N$ are fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions:

(i) $\mathcal{M}(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$;
(ii) $\mathcal{M}(x, y, 0) = 0$ for all $x, y \in X$;
(iii) $\mathcal{M}(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;
(iv) $\mathcal{M}(x, y, t) = \mathcal{M}(y, x, t)$ for all $x, y \in X$ and $t > 0$;
(v) $\mathcal{M}(x, y, t) * \mathcal{M}(y, z, s) \leq \mathcal{M}(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;
(vi) $\mathcal{M}(x, y, .): (0, \infty) \longrightarrow [0,1]$ is left continuous for all $x, y \in X$;
(vii) $\lim_{t\to\infty} \mathcal{M}(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$;
(viii) $N(x, y, 0) = 1$ for all $x, y \in X$;
(ix) $N(x, y, t) = 0$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;
(x) $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$;
(xi) $N(x, y, t) \lozenge N(y, z, s) \geq N(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;
(xii) $N(x, y, .): (0, \infty) \longrightarrow [0,1]$ is right continuous for all $x, y \in X$;
(xiii) $\lim_{t\to\infty} N(x, y, t) = 0$ for all $x, y \in X$;

Then $(\mathcal{M}, N)$ is called an intuitionistic fuzzy metric on X. The functions $\mathcal{M}(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness between x and y with respect to t, respectively.

Example 2.1: Let $X = \mathbb{N}$. Define $a * b = max\{0, a + b - 1\}$ and

$a \lozenge b = a + b - ab$ for all $a, b \in [0,1]$ and let $\mathcal{M}$ and N be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$\mathcal{M}(x, y, t) = \begin{cases} \frac{x}{y}, & if\ x \leq y \\ \frac{y}{x}, & if\ y \leq x \end{cases}$$

$$N(x, y, t) = \begin{cases} \frac{y - x}{y}, & if\ x \leq y \\ \frac{x - y}{x}, & if\ y \leq x \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then $(X, \mathcal{M}, N, *, \lozenge)$ is an intuitionistic fuzzy metric space.

Definition 2.4: An intuitionistic fuzzy set $\mathcal{A}_{\zeta,\eta}$ in a universe U is an object $\mathcal{A}_{\zeta,\eta} = \{(\zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) | u \in U\}$, where, for all $u \in U$, $\zeta_{\mathcal{A}}(u) \in [0, 1]$ and $\eta_{\mathcal{A}}(u) \in [0, 1]$ are called the membership degree and the non−membership degree, respectively of u in $\mathcal{A}_{\zeta,\eta}$ and furthermore they satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

Remark 2.1: An intuitionistic fuzzy metric spaces with continuous t $-$norm $*$ and continuous t $-$conorm $\Diamond$ defined by $a * a \geq a$ and $(1-a) \Diamond (1-a) \leq (1-a)$ for all $a \in [0,1]$. Then for all $x, y \in X, \mathcal{M}(x, y,)$ is non decreasing and $N(x, y, \Diamond)$ is non increasing.

Remark 2.2: Inintuitionistic fuzzy metric space X, $\mathcal{M}(x, y, .)$ is non decreasing and $N(x, y, .)$ is non-increasing for all $x, y \in X$. Alaca, Turkoglu and Yildiz [2] introduced the following notions:

Definition 2.5: Let $(X, \mathcal{M}, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. Then

(i) A sequence $\{x_n\}$ in X is said to be Cauchy sequence if for all $t > 0$, and $p > 0$,
$$\lim_{n\to\infty} \mathcal{M}\left(x_{n+p}, x_n, t\right) = 1, \lim_{n\to\infty} N\left(x_{n+p}, x_n, t\right) = 0.$$
(ii) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for all $t > 0, \lim_{n\to\infty} \mathcal{M}\,(x_n, x, t) = 1, \lim_{n\to\infty} N\,(x_n, x, t) = 0.$

Since $*$ and $\Diamond$ are continuous, the limit is uniquely determined from (v) and (xi) of definition 2.3, respectively.

Definition 2.6: An intuitionistic fuzzy metric space $(X, \mathcal{M}, N, *, \Diamond)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Definition 2.7: An intuitionistic fuzzy metric space $(X, \mathcal{M}, N, *, \Diamond)$ is said to be compact if every sequence in X contains a convergent subsequence.

Lemma 2.1: Let $(X, \mathcal{M}, N, *, \Diamond)$ be an intuitionistic fuzzy metric space and for all $x, y \in X, t > 0$ and if for a number $k \in (0, 1)$,

$\mathcal{M}(x, y, kt) \geq \mathcal{M}(x, y, t)$ and $N(x, y, kt) \leq N(x, y, t)$ then $x = y$.

Lemma 2.2: Let $(X, \mathcal{M}, N, *, \Diamond)$ be an intuitionistic fuzzy metric space and

$\{y_n\}$ be a sequence in X. If there exists a number $k \in (0, 1)$such that

$$\mathcal{M}(y_{n+2}, y_{n+1}, kt) \geq \mathcal{M}(y_{n+1}, y_n, t),$$

$$N(y_{n+2}, y_{n+1}, kt) \leq N(y_{n+1}, y_n, t)$$ for all $t > 0$ and $n = 1, 2, \ldots$, then $\{y_n\}$ is a Cauchy sequence in X.

Definition 2.8: A pair of self mappings (f, g) of an intuitionistic fuzzy metric space $(X, \mathcal{M}, N, *, \Diamond)$ is said to be compatible

if $\lim_{n\to\infty} \mathcal{M}\,(fgx_n, gfx_n, t) = 1$, and $\lim_{n\to\infty} N\,(fgx_n, gfx_n, t) = 0$ for every $t > 0$, whenever $\{x_n\}$ is a sequence in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

Definition 2.9: A pair of self mappings (f, g) of an intuitionistic fuzzy metric space $(X, \mathcal{M}, N, *, \Diamond)$ is said to be non compatible if $\lim_{n\to\infty} \mathcal{M}\,(fgx_n, gfx_n, t) \neq 1$, or non-existent and $\lim_{n\to\infty} N\,(fgx_n, gfx_n, t) \neq 0$ or non$-$existent for every $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$. Jungck G. and Rhoades [11] introduced the concept of weakly compatible maps as follows:

Definition 2.10: Two self maps f and g are said to be weakly compatible if they commute at coincidence points.

Definition 2.11: Let $(X, \mathcal{M}, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. A finite sequence $x = x_0, x_1, x_2, \cdots, x_n = y$ is called $\in -$chain from x to y if there exists a positive number $\in > 0$ such that $\mathcal{M}(x_i, x_{i-1}, t) > 1-\in$ and $N(x_i, x_{i-1}, t) > 1-\in$ for all $t > 0$ and $i = 1,2, \cdots, n.$ An intuitionistic fuzzy metric space $(X, \mathcal{M}, N, *, \Diamond)$ is called $\in -$chainable if for any $x, y \in X,$ there exists an $\in -$chain from x to y.

3. The Main Results

Theorem 3.1: Let $A, B, S,$ and T be self maps of a complete $\in -$chainable intuitionistic fuzzy metric spaces $(X, \mathcal{M}, N, *, \Diamond)$ with continuous $t -$norm $*$ and continuous $t -$conorm $\Diamond$ defined by $a * a \geq a$ and

$(1 - a) \Diamond (1 - a) \leq (1 - a)$ for all $a \in [0,1]$. Satisfying the following condition:

(1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X),$
(2) A and S are continuous,
(3) The pairs $(A, S)$and (B, T) are weakly compatible,
(4) There exist $q \in (0,1)$ such that

$$\begin{aligned}\mathcal{M}(Ax, By, qt) & \\ \geq \Big\{ & \mathcal{M}(Sx, Ty, t) * \mathcal{M}(Sx, Ax, t) * \frac{1}{2}[\mathcal{M}(Sx, Ty, t) + \mathcal{M}(Ax, Ty, t)] \\ & * \frac{1}{2}[\mathcal{M}(Ax, By, t) + \mathcal{M}(Sx, By, t)] * \mathcal{M}(By, Ty, t)\Big\}\end{aligned}$$

and

$$\begin{aligned}N(Ax, By, qt) \leq \Big\{ & N(Sx, Ty, t) \Diamond N(Sx, Ax, t) \Diamond \frac{1}{2}[N(Sx, Ty, t) + N(Ax, Ty, t)] \\ & \Diamond \frac{1}{2}[N(Ax, By, t) + N(Sx, By, t)] \Diamond N(By, Ty, t)\Big\}.\end{aligned}$$

For every $x, y \in X$ and $t > 0.$ Then A, B, S and T have a unique common fixed point in $X.$

Proof: As $A(X) \subseteq T(X),$ for any $x_0 \in X,$ there exist a point $x_1 \in X$ such that $Ax_0 = Tx_1.$ Since $B(X) \subseteq S(X),$ for this point $x_1,$ we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2.$ Inductively, we can find a sequence $\{y_n\}$ in X as follows:

$y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for $n = 1,2, \cdots$

By Theorem of Alaca et al. [1], we can conclude that $\{y_n\}$ in $X.$

Since X is complete, therefore sequence $\{y_n\}$ in X converges to z for some z in X and so the sequences $\{Tx_{2n-1}\}, \{Ax_{2n-2}\}, \{Sx_{2n}\}$ and $\{Bx_{2n-1}\}$ also converges to $z.$

Since X is $\in -$chainable, there exists $\in -$chain from x_n to x_{n+1}, that is there exists a finite sequence

$$x_n = y_1, y_2, \cdots, y_l = x_{n+1}$$

such that $\mathcal{M}(y_i, y_{i-1}, t) > 1-\in$

and $N(y_i, y_{i-1}, t) < 1-\in$ for all $t > 0$ and $i = 1,2,\cdots,l$.

So we have

$$\mathcal{M}(x_n, x_{n+1}, t) \geq \mathcal{M}\left(y_1, y_2, \frac{t}{l}\right) * \mathcal{M}\left(y_2, y_3, \frac{t}{l}\right) * \cdots *$$

$$\mathcal{M}\left(y_{l-1}, y_l, \frac{t}{l}\right) > (1-\in) * (1-\in) * \cdots * (1-\in) \geq (1-\in)$$

and

$$N(x_n, x_{n+1}, t) \leq N\left(y_1, y_2, \frac{t}{l}\right) \lozenge N\left(y_2, y_3, \frac{t}{l}\right) \lozenge \cdots \lozenge$$

$$N\left(y_{l-1}, y_l, \frac{t}{l}\right) < (1-\in) \lozenge (1-\in) \lozenge \cdots \lozenge (1-\in) \leq (1-\in)$$

For $m > n$,

$$\mathcal{M}\ (x_n, x_m, t)\ \geq\ \mathcal{M}(x_n, x_{n+1}, \frac{t}{m-n})\ * \mathcal{M}(x_{n+1}, x_{n+2}, \frac{t}{m-n})$$

$$* \cdots * \mathcal{M}\left(x_{m-1}, x_m, \frac{t}{m-n}\right) > (1-\in) * (1-\in) * \cdots * (1-\in) \geq (1-\in)$$ and

$$N\ (x_n, x_m, t)\ \leq\ N(x_n, x_{n+1}, \frac{t}{m-n}) \lozenge N(x_{n+1}, x_{n+2}, \frac{t}{m-n})$$

$$\lozenge \cdots \lozenge \mathcal{M}\left(x_{m-1}, x_m, \frac{t}{m-n}\right) < (1-\in)$$

$$\lozenge (1-\in) \lozenge \cdots \lozenge (1-\in) \leq (1-\in)$$

Therefore $\{x_n\}$ is a Cauchy sequence in X and hence there exists x in X such that $x_n \to x$. from condition (2) $Ax_{2n-2} \to Ax,\ Sx_{2n} \to Sx$ as limit $n \to \infty$. By uniqueness of limits, we have $Ax = z = Sx$. Since pair (A, S) is weakly compatible, therefore, $ASx = SAx$ and so $Az = Sz$, from condition (2) we have $ASx_{2n} \to ASx$ and therefore, $ASx_{2n} \to Sz$. Also from continuity of S, we have $SSx_{2n} \to Sz$.

from condition (4), we get

$$\begin{aligned}\mathcal{M}(ASx_{2n}, Bx_{2n-1}, qt) \\ \geq \Big\{&\mathcal{M}(SSx_{2n}, Tx_{2n-1}, t) * \mathcal{M}(SSx_{2n}, ASx_{2n}, t) \\ &* \frac{1}{2}[\mathcal{M}(SSx_{2n}, Tx_{2n-1}, t) + \mathcal{M}(ASx_{2n}, Tx_{2n-1}, t)] \\ &* \frac{1}{2}[\mathcal{M}(ASx_{2n}, Bx_{2n-1}, t) + \mathcal{M}(SSx_{2n}, Bx_{2n-1}, t)] * \mathcal{M}(Bx_{2n-1}, Tx_{2n-1}, t)\Big\}\end{aligned}$$

and

$$N(ASx_{2n}, Bx_{2n-1}, qt) \le \Big\{N(SSx_{2n}, Tx_{2n-1}, t) \lozenge N(SSx_{2n}, ASx_{2n}, t) \lozenge \frac{1}{2}[N(SSx_{2n}, Tx_{2n-1}, t) + N(ASx_{2n}, Tx_{2n-1}, t)] \lozenge \frac{1}{2}[N(ASx_{2n}, Bx_{2n-1}, t) + N(SSx_{2n}, Bx_{2n-1}, t)] \lozenge N(Bx_{2n-1}, Tx_{2n-1}, t)\Big\}.$$

Proceeding limit as $n \to \infty$, we have

$$\mathcal{M}(Sz, z, qt) \ge \Big\{\mathcal{M}(Sz, z, t) * \mathcal{M}(Sz, Sz, t) * \frac{1}{2}[\mathcal{M}(Sz, z, t) + \mathcal{M}(Sz, z, t)] * \frac{1}{2}[\mathcal{M}(Sz, z, t) + \mathcal{M}(Sz, z, t)] * \mathcal{M}(z, z, t)\Big\}$$

and

$$N(Sz, z, qt) \le \Big\{N(Sz, z, t) \lozenge N(Sz, Sz, t) \lozenge \frac{1}{2}[N(Sz, z, t) + N(Sz, z, t)] \lozenge \frac{1}{2}[N(Sz, z, t) + N(Sz, z, t)] \lozenge N(z, z, t)\Big\}.$$

From lemma 2.1, we get $Sz = z$, and hence $Az = Sz = z$.

Since $A(X) \subseteq T(X)$, there exists v in X such that $Tv = Az = z$.

from condition (4), we have

$$\mathcal{M}(Ax_{2n}, Bv, qt) \ge \Big\{\mathcal{M}(Sx_{2n}, Tv, t) * \mathcal{M}(Sx_{2n}, Ax_{2n}, t) * \frac{1}{2}[\mathcal{M}(Sx_{2n}, Tv, t) + \mathcal{M}(Ax_{2n}, Tv, t)] * \frac{1}{2}[\mathcal{M}(Ax_{2n}, Bv, t) + \mathcal{M}(Sx_{2n}, Bv, t)] * \mathcal{M}(Bv, Tv, t)\Big\}$$

and

$$N(Ax_{2n}, Bv, qt) \le \Big\{N(Sx_{2n}, Tv, t) \lozenge N(Sx_{2n}, Ax_{2n}, t) \lozenge \frac{1}{2}[N(Sx_{2n}, Tv, t) + N(Ax_{2n}, Tv, t)] \lozenge \frac{1}{2}[N(Ax_{2n}, Bv, t) + N(Sx_{2n}, Bv, t)] \lozenge N(Bv, Tv, t)\Big\}.$$

Letting $n \to \infty$, we have

$$\mathcal{M}(z, Bv, qt) \ge \Big\{\mathcal{M}(z, Tv, t) * \mathcal{M}(z, z, t) * \frac{1}{2}[\mathcal{M}(z, Tv, t) + \mathcal{M}(z, Tv, t)] * \frac{1}{2}[\mathcal{M}(z, Bv, t) + \mathcal{M}(z, Bv, t)] * \mathcal{M}(Bv, Tv, t)\Big\}$$

$$= \Big\{\mathcal{M}(z,z,t) * \mathcal{M}(z,z,t) * \frac{1}{2}[\mathcal{M}(z,z,t) + \mathcal{M}(z,z,t)] * \frac{1}{2}[\mathcal{M}(z,Bv,t) + \mathcal{M}(z,Bv,t)] * \mathcal{M}(Bv,z,t) * \mathcal{M}(z,z,t) * \mathcal{M}(z,Bv,t)\Big\} \geq \mathcal{M}(Bv,z,t)$$

and

$$N(z,Bv,qt) \leq \Big\{N(z,Tv,t) \lozenge N(z,z,t) \lozenge \frac{1}{2}[N(z,Tv,t) + N(z,Tv,t)] \lozenge \frac{1}{2}[N(z,Bv,t) + N(z,Bv,t)] \lozenge N(Bv,Tv,t)\Big\}$$

$$= \Big\{N(z,z,t) \lozenge N(z,z,t) \lozenge \frac{1}{2}[N(z,z,t) + N(z,z,t)] \lozenge \frac{1}{2}[N(z,Bv,t) + N(z,Bv,t)] \lozenge N(Bv,z,t)\Big\} \leq N(Bv,z,t).$$

By lemma 2.1, we have $Bv = z$, and therefore, we have $Tv = Bv = z$.

Since (B,T) is weakly compatible, therefore, $TBv = BTv$ and

Hence $Tz = Bz$. from condition (4), we have

$$\mathcal{M}(Ax_{2n},Bz,qt) \geq \Big\{\mathcal{M}(Sx_{2n},Tz,t) * \mathcal{M}(Sx_{2n},Ax_{2n},t) * \frac{1}{2}[\mathcal{M}(Sx_{2n},Tz,t) + \mathcal{M}(Ax_{2n},Tz,t)] * \frac{1}{2}[\mathcal{M}(Ax_{2n},Bz,t) + \mathcal{M}(Sx_{2n},Bz,t)] * \mathcal{M}(Bz,Tz,t)\Big\}$$

and

$$N(Ax_{2n},Bz,qt) \leq \Big\{N(Sx_{2n},Tz,t) \lozenge N(Sx_{2n},Ax_{2n},t) \lozenge \frac{1}{2}[N(Sx_{2n},Tz,t) + N(Ax_{2n},Tz,t)] \lozenge \frac{1}{2}[N(Ax_{2n},Bz,t) + N(Sx_{2n},Bz,t)] \lozenge N(Bz,Tz,t)\Big\}.$$

Letting $n \to \infty$, we have

$$\mathcal{M}(z,Bz,qt) \geq \Big\{\mathcal{M}(z,Tz,t) * \mathcal{M}(z,z,t) * \frac{1}{2}[\mathcal{M}(z,Tz,t) + \mathcal{M}(z,Tz,t)] * \frac{1}{2}[\mathcal{M}(z,Bz,t) + \mathcal{M}(z,Bz,t)] * \mathcal{M}(Bz,Tz,t)\Big\}$$

$$= \Big\{\mathcal{M}(z,z,t) * \mathcal{M}(z,z,t) * \frac{1}{2}[\mathcal{M}(z,z,t) + \mathcal{M}(z,z,t)] * \frac{1}{2}[\mathcal{M}(z,Bz,t) + \mathcal{M}(z,Bz,t)] * \mathcal{M}(Bz,z,t)\Big\} \geq \mathcal{M}(Bz,z,t)$$

and

$$N(z,Bz,qt) \leq \left\{N(z,Tz,t) \lozenge N(z,z,t) \lozenge \frac{1}{2}[N(z,Tz,t) + N(z,Tz,t)] \lozenge \frac{1}{2}[N(z,Bz,t) + N(z,Bz,t)] \lozenge N(Bz,Tz,t)\right\}$$

$$= \left\{N(z,z,t) \lozenge N(z,z,t) \lozenge \frac{1}{2}[N(z,z,t) + N(z,z,t)] \lozenge \frac{1}{2}[N(z,Bz,t) + N(z,Bz,t)] \lozenge N(Bz,z,t)\right\} \leq N(Bz,z,t).$$

Which implies that$Bz = z$. Therefore, $Az = Sz = Bz = Tz = z$.

Hence A, B, S and T have a unique common fixed point in X.

For uniqueness, let w be another common fixed point of A, B, S and T.

Then from condition (4), we have

$$\mathcal{M}(z,w,qt) = \mathcal{M}(Az,Bw,qt) \geq \left\{\mathcal{M}(Sz,Tw,t) * \mathcal{M}(Sz,Az,t) * \frac{1}{2}[\mathcal{M}(Sz,Tw,t) + \mathcal{M}(Az,Tw,t)] * \frac{1}{2}[\mathcal{M}(Az,Bw,t) + \mathcal{M}(Sz,Bw,t)] * \mathcal{M}(Bw,Tw,t)\right\} \geq \mathcal{M}(z,w,t)$$

and

$$N(z,w,qt) = N(Az,Bw,qt) \leq \left\{N(Sz,Tw,t) \lozenge N(Sz,Az,t) \lozenge \frac{1}{2}[N(Sz,Tw,t) + N(Az,Tw,t)] \lozenge \frac{1}{2}[N(Az,Bw,t) + N(Sz,Bw,t)] \lozenge N(Bw,Tw,t)\right\} \leq N(z,w,t).$$

From lemma 2.1, we conclude that $z = w$.

Hence A, B, S and T have a unique common fixed point in X.

Theorem 2.2: Let A, B, S, and T be self maps of a complete $\in -$chainable intuitionistic fuzzy metric spaces $(X, \mathcal{M}, N, *, \lozenge)$ with continuous $t -$norm $*$ and continuous $t -$conorm $\lozenge$ defined by $a * a \geq a$ and $(1 - a) \lozenge (1 - a) \leq (1 - a)$ for all $a \in [0,1]$. Satisfying the following condition:

(1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$,
(2) A and S are continuous,
(3) The pairs (A, S) and (B, T) are weakly compatible,
(4) There exist $q \in (0,1)$ such that

$$\mathcal{M}(Ax,By,qt) \geq \{\mathcal{M}(Sx,Ty,t) * \mathcal{M}(Sx,Ax,t) * \mathcal{M}(By,Ty,t) * \mathcal{M}(Ax,Ty,t) * \mathcal{M}(Sx,By,t)\}$$

and

$$N(Ax, By, qt) \leq \{N(Sx, Ty, t) \lozenge N(Sx, Ax, t) \lozenge N(By, Ty, t) \lozenge N(Ax, Ty, t) \lozenge N(Sx, By, t)\}$$

For every $x, y \in X$ and $t > 0 >$. Then A, B, S and T have a unique common fixed point in X.

Proof: As $A(X) \subseteq T(X)$, for any $x_0 \in X$, there exist a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subseteq S(X)$, for this point x_1, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$. Inductively, we can find a sequence $\{y_n\}$ in X as follows:

$y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for $n = 1,2,\cdots$

By Theorem of Alaca et al. [1], we can conclude that $\{y_n\}$ in X.

Since X is complete, therefore sequence $\{y_n\}$ in X converges to z for some z in X and so the sequences $\{Tx_{2n-1}\}, \{Ax_{2n-2}\}, \{Sx_{2n}\}$ and $\{Bx_{2n-1}\}$ also converges to z. Since X is $\in -$chainable, there exists $\in -$chain from x_n to x_{n+1}, that is there exists a finite sequence $x_n = y_1, y_2, \cdots, y_l = x_{n+1}$

such that $\mathcal{M}(y_i, y_{i-1}, t) > 1-\in$

and $N(y_i, y_{i-1}, t) < 1-\in$ for all $t > 0$ and $i = 1,2, \cdots, l$.

So we have

$$\mathcal{M}(x_n, x_{n+1}, t) \geq \mathcal{M}\left(y_1, y_2, \frac{t}{l}\right) * \mathcal{M}\left(y_2, y_3, \frac{t}{l}\right) * \cdots *$$

$$\mathcal{M}\left(y_{l-1}, y_l, \frac{t}{l}\right) > (1-\in) * (1-\in) * \cdots * (1-\in) \geq (1-\in)$$

and

$$N(x_n, x_{n+1}, t) \leq N\left(y_1, y_2, \frac{t}{l}\right) \lozenge N\left(y_2, y_3, \frac{t}{l}\right) \lozenge \cdots \lozenge$$

$$N\left(y_{l-1}, y_l, \frac{t}{l}\right) < (1-\in) \lozenge (1-\in) \lozenge \cdots \lozenge (1-\in) \leq (1-\in)$$

For $m > n$,

$$\mathcal{M}(x_n, x_m, t) \geq \mathcal{M}(x_n, x_{n+1}, \frac{t}{m-n}) * \mathcal{M}(x_{n+1}, x_{n+2}, \frac{t}{m-n})$$

$$* \cdots * \mathcal{M}(x_{m-1}, x_m, \frac{t}{m-n}) > (1-\in) * (1-\in) * \cdots * (1-\in) \geq (1-\in)$$ and

$$N(x_n, x_m, t) \leq N(x_n, x_{n+1}, \frac{t}{m-n}) \lozenge N(x_{n+1}, x_{n+2}, \frac{t}{m-n})$$

$$\lozenge \cdots \lozenge \mathcal{M}(x_{m-1}, x_m, \frac{t}{m-n}) < (1-\in) \lozenge (1-\in) \lozenge \cdots \lozenge (1-\in) \leq (1-\in)$$

Therefore $\{x_n\}$ is a Cauchy sequence in X and hence there exists x in X such that $x_n \to x$. from condition (2) $Ax_{2n-2} \to Ax,\ Sx_{2n} \to Sx$ as limit $n \to \infty$. By uniqueness of limits, we have $Ax = z = Sx$. Since pair (A, S) is weakly compatible, therefore, $ASx = SAx$ and so $Az = Sz$.

from condition (2) we have $ASx_{2n} \to ASx$ and therefore, $ASx_{2n} \to Sz$. Also from continuity of S, we have $SSx_{2n} \to Sz$. from condition (4), we get

$$\begin{aligned}\mathcal{M}(ASx_{2n}, Bx_{2n-1}, qt) &\\ &\geq \{\mathcal{M}(SSx_{2n}, Tx_{2n-1}, t) * \mathcal{M}(SSx_{2n}, ASx_{2n}, t) * \mathcal{M}(Bx_{2n-1}, Tx_{2n-1}, t)\\ &* \mathcal{M}(ASx_{2n}, Tx_{2n-1}, t) * \mathcal{M}(SSx_{2n}, Bx_{2n-1}, t)\}\end{aligned}$$

and

$$\begin{aligned}N(ASx_{2n}, Bx_{2n-1}, qt) &\\ &\leq \{N(SSx_{2n}, Tx_{2n-1}, t) \lozenge N(SSx_{2n}, ASx_{2n}, t) \lozenge N(Bx_{2n-1}, Tx_{2n-1}, t)\\ &\lozenge N(ASx_{2n}, Tx_{2n-1}, t) \lozenge N(SSx_{2n}, Bx_{2n-1}, t)\}\end{aligned}$$

Proceeding limit as $n \to \infty$, we have

$$\mathcal{M}(Sz, z, qt) \geq \{\mathcal{M}(Sz, z, t) * \mathcal{M}(Sz, Sz, t) * \mathcal{M}(z, z, t) * \mathcal{M}(Sz, z, t) * \mathcal{M}(Sz, z, t)\}$$

and

$$N(Sz, z, qt) \leq \{N(Sz, z, t) \lozenge N(Sz, Sz, t) \lozenge N(z, z, t) \lozenge N(Sz, z, t) \lozenge N(Sz, z, t)\}.$$

From lemma 2.1, we get $Sz = z$, and hence $Az = Sz = z$.

Since $A(X) \subseteq T(X)$, there exists v in X such that $Tv = Az = z$.

from condition (4), we have

$$\begin{aligned}\mathcal{M}(Ax_{2n}, Bv, qt) &\\ &\geq \{\mathcal{M}(Sx_{2n}, Tv, t) * \mathcal{M}(Sx_{2n}, Ax_{2n}, t) * \mathcal{M}(Bv, Tv, t) * \mathcal{M}(Ax_{2n}, Tv, t)\\ &* \mathcal{M}(Sx_{2n}, Bv, t)\}\end{aligned}$$

and

$$\begin{aligned}N(Ax_{2n}, Bv, qt) &\\ &\leq \{N(Sx_{2n}, Tv, t) \lozenge N(Sx_{2n}, Ax_{2n}, t) \lozenge N(Bv, Tv, t) \lozenge N(Ax_{2n}, Tv, t)\\ &\lozenge N(Sx_{2n}, Bv, t)\}\end{aligned}$$

Letting $n \to \infty$, we have

$$\mathcal{M}(z, Bv, qt) \geq \{\mathcal{M}(z, Tv, t) * \mathcal{M}(z, z, t) * \mathcal{M}(Bv, Tv, t) * \mathcal{M}(z, Tv, t) * \mathcal{M}(z, Bv, t)\}$$

$$= \{\mathcal{M}(z, z, t) * \mathcal{M}(z, z, t) * \mathcal{M}(Bv, z, t) * \mathcal{M}(z, z, t) * \mathcal{M}(z, Bv, t)\} \geq \mathcal{M}(Bv, z, t)$$

and

$$N(z, Bv, qt) \leq \{N(z, Tv, t) \lozenge N(z, z, t) \lozenge N(Bv, Tv, t) \lozenge N(z, Tv, t) \lozenge N(z, Bv, t)\}$$

$$= \{N(z, z, t) \lozenge N(z, z, t) \lozenge N(Bv, z, t) \lozenge N(z, z, t) \lozenge N(z, Bv, t)\} \leq N(Bv, z, t).$$

By lemma 2.1, we have $Bv = z$, and therefore, we have $Tv = Bv = z$.

Since (B,T) is weakly compatible, therefore, $TBv = BTv$ and

Hence $Tz = Bz.$ from condition (4), we have

$$\mathcal{M}(Ax_{2n}, Bz, qt) \geq \{\mathcal{M}(Sx_{2n}, Tz, t) * \mathcal{M}(Sx_{2n}, Ax_{2n}, t) * \mathcal{M}(Bz, Tz, t) * \mathcal{M}(Ax_{2n}, Tz, t) * \mathcal{M}(Sx_{2n}, Bz, t)\}$$

and

$$N(Ax_{2n}, Bz, qt) \leq \{N(Sx_{2n}, Tz, t) \lozenge N(Sx_{2n}, Ax_{2n}, t) \lozenge N(Bz, Tz, t) \lozenge N(Ax_{2n}, Tz, t) \lozenge N(Sx_{2n}, Bz, t)\}.$$

Letting $n \to \infty$, we have

$$\mathcal{M}(z, Bz, qt) \geq \{\mathcal{M}(z, Tz, t) * \mathcal{M}(z, z, t) * \mathcal{M}(Bz, Tz, t) * \mathcal{M}(z, Tz, t) * \mathcal{M}(z, Bz, t)\}$$

$$= \{\mathcal{M}(z, z, t) * \mathcal{M}(z, z, t) * \mathcal{M}(Bz, z, t) * \mathcal{M}(z, z, t) * \mathcal{M}(z, Bz, t)\} \geq \mathcal{M}(Bz, z, t)$$

and

$$N(z, Bz, qt) \leq \{N(z, Tz, t) \lozenge N(z, z, t) \lozenge N(Bz, Tz, t) \lozenge N(z, Tz, t) \lozenge N(z, Bz, t)\}$$

$$= \{N(z, z, t) \lozenge N(z, z, t) \lozenge N(Bz, z, t) \lozenge N(z, z, t) \lozenge N(z, Bz, t)\} \leq N(Bz, z, t).$$

Which implies that $Bz = z$.

Therefore, $Az = Sz = Bz = Tz = z$.

Hence A, B, S and T have a unique common fixed point in X.

For uniqueness, let w be another common fixed point of A, B, S and T.

Then from condition (4), we have

$$\mathcal{M}(z, w, qt) = \mathcal{M}(Az, Bw, qt) \geq \{\mathcal{M}(Sz, Tw, t) * \mathcal{M}(Sz, Az, t) * \mathcal{M}(Bw, Tw, t) * \mathcal{M}(Az, Tw, t) * \mathcal{M}(Sz, Bw, t)\} \geq \mathcal{M}(z, w, t)$$

and

$$N(z, w, qt) = N(Az, Bw, qt) \leq \{N(Sz, Tw, t) \lozenge N(Sz, Az, t) \lozenge N(Bw, Tw, t) \lozenge N(Az, Tw, t) \lozenge N(Sz, Bw, t)\} \leq N(z, w, t).$$

From lemma 2.1, we conclude that $z = w$.

Hence A, B, S and T have a unique common fixed point in X.

4. Conclusion

In this chapter, we give some conditions of which four self─mappings for Intuitionistic fuzzy metric spaces have a unique common fixed point. This work can be easily extended by increasing the number of self─mappings and establishing the fixed point theorems in more generalized settings

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