FROM GEOMETRIC AVERAGE TO ARITHMETIC AVERAGE AND QUADRATIC AVERAGE

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ABSTRACT
In mathematics, functional equations are very difficult and arise all areas of mathematics, even more, science, engineering, and social sciences. They appear at all levels of mathematics. The theory of functional equations were born very early. Many authors studied functional equations. In this artical, we study some problems about geometric average, arithmetic average and quadratic average.

Keyword: *Functional equalities, geometric average, arithmetic average, quadratic average.*

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1. PRELIMINARIES

In this artical, we would like to look at some expressions

$$\sqrt{xy}, x, y \in R^+;$$
$$\frac{f(x)+f(y)}{2}, x, y \in R^+;$$
$$\sqrt{\frac{\left[f(x)\right]^2+\left[f(y)\right]^2}{2}} x, y \in R^+.$$

In this paper, we use method of substitution

+) Example, let $x = u$ such that $f(u)$ appears much in the equation.

+) Let $x = u, y = v$ interchange to refer $f(u)$ and $f(v)$.

+) Let $f(0) = c, f(1) = c$.

+) If f is surjection, exist $c : f(c) = 0$. Choice x, y to destroy $f(g(x, y))$ in the equation. The function has x, we show that it is injective or surjection.

+) To occur $f(x)$.

+) $f(x) = f(y)$ for all x, y ∈ A. Hence $f(x) = const$ for all $x \in A$.

On the other hand, we use math induction and the continuos of function.

2. FROM GEOMETRIC AVERAGE TO ARITHMETIC AVERAGE AND QUADRATIC AVERAGE

In this part, we would like to look at some problems from geometric average to arithmetic average and quadratic average.

2.1. Problem 1 (From arithmetic mean to arithmetic mean). Determine all functions $f(x)$ continuous on R such that:

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}; \forall x, y \in R. \quad (1)$$

Solution. Let $f(0) = c$ and $f(x) = c + g(x)$. Then $g(0) = 0$. Replacing (1), we get

$$c+g\left(\frac{x+y}{2}\right)=\frac{2c+g(x)+g(y)}{2};\forall x,y\in R.$$
$$\Leftrightarrow g\left(\frac{x+y}{2}\right)=\frac{g(x)+g(y)}{2};\forall x,y\in R.\ (2)$$

With $g(0)=0$.
Let $y=0$. Then (2), we have

$$g\left(\frac{x}{2}\right)=\frac{g(x)}{2},$$

or

$$g\left(\frac{x+y}{2}\right)=\frac{g(x)+g(y)}{2};\forall x,y\in R.$$

Replacing (2), we have

$$g\left(\frac{x+y}{2}\right)=\frac{g(x)+g(y)}{2};\forall x,y\in R.$$
$$\Leftrightarrow g(x+y)=g(x)+g(y);x,y\in R.\quad(3)$$

Since $g(x)$ is continuous on R. Then (3) is Cauchy function. Then $g(x)=ax$.

We get $f(x)$=ax+c, a, c $\in$ R.

We can check directly $f(x)$=ax+c, a, c $\in$ R. satisfies (1).

Hence,

$$f(x)=ax+c,\ a,\ c\in R.$$

2.2. Problem 2. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ which are continuous on $\mathbb{R}$ and satisfy the equation

$$f\left(\sqrt{xy}\right)=\frac{f(x)+f(y)}{2},\forall x,y\in R^{+}.\quad(4)$$

Solution. Because $x>0, y>0$ then we set $x=e^{u}, y=e^{v}, u,v\in R$. From (4), we have

$$f\left(\sqrt{e^{u+v}}\right)=\frac{f(e^{u})+f(e^{v})}{2},\forall u,v\in R$$
$$\Leftrightarrow f\left(e^{\frac{u+v}{2}}\right)=\frac{f(e^{u})+f(e^{v})}{2},\forall u,v\in R$$

Setting $g(t)=f(e^{t}),\forall t\in R,$ we have

$$g\left(\frac{u+v}{2}\right)=\frac{g(u)+g(v)}{2},\forall u,v\in R.$$

By Problem 1, we have

$$g(t)=at+c;\forall t\in R.$$

Hence,

$$f(t) = a\ln t + c; a, c \in R.$$

So,

$$f(x) = a\ln x + c; a, c \in R.$$

We can check directly $f(x) = a\ln x + c; a, c \in R$ satisfies (4).

There for,

$$f(x) = a\ln x + c; a, c \in R.$$

2.3. Problem 3. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ which are continuous on the real axis and satisfy the equation

$$f\left(\sqrt{xy}\right) = \frac{\sqrt{\left[f(x)\right]^2 + \left[f(y)\right]^2}}{2}, \forall x, y \in R^2. \quad (5)$$

Solution. By assumption, we have $f(x) \geq 0, \forall x \geq 0.$ Because $x > 0, y > 0$ then we set $x = e^u, y = e^v, u, v \in R$. From (5), we have

$$f\left(\sqrt{e^{u+v}}\right) = \sqrt{\frac{\left[f\left(e^u\right)\right]^2 + \left[f\left(e^v\right)\right]^2}{2}}, \forall u, v \in R$$

$$\Leftrightarrow \left[f\left(e^{\frac{u+v}{2}}\right)\right]^2 = \frac{\left[f\left(e^u\right)\right]^2 + \left[f\left(e^v\right)\right]^2}{2}, \forall u, v \in R$$

Setting $g(t) = \left[f\left(e^t\right)\right]^2, \forall t \in R,$ we have

$$g\left(\frac{u+v}{2}\right) = \frac{g(u) + g(v)}{2}, \forall u, v \in R.$$

By Problem 1, we have

$$g(t) = at + c; \forall t \in R.$$

Because $g(t) \geq 0, \forall t \in R$, we choise $a = 0, c \geq 0.$ Then

$$f(x) \equiv b, b \geq 0.$$

We can check directly $f(x) \equiv b, b \geq 0$ satisfies (5).

There for,

$$f(x) \equiv b, b \geq 0.$$

3. CONCLUSIONS

In this paper, we establish some problems from geometric average to arithmetic average and quadratic average.
It is very good for teachers and students.

4. REFERENCES

[1]. Titu Andresscu, Iurie Boreico (2007). Functional equations – 17 Chapters and 199 Problems with solution, Electronic Edition.

[2]. T. Aczel (1966). Lectures on functional equations and their applications, Academic Press, New York/San Francisco/London