FUZZY NORMED AND BANACH SPACE IN FUZZY TOPOLOGICAL SPACES

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Abstract

The main aim of this paper is to consider the fuzzy normed spaces and define the fuzzy Banach spaces and also the introduction of fuzzy metric spaces of its quotients and prove some theorems and lemma's with the example. Based on the open mapping and closed graph theorems on these fuzzy metric spaces.

Keyword

Fuzzy space, Fuzzy normed space, Fuzzy Banach space, Fuzzy metric space.

1. Introduction

Many mathematicians have studied fuzzy normed spaces from several angles. The theory of fuzzy sets was introduced by L. Zadeh in 1965. The concept of fuzzy norm was introduced by Katsaras in 1984. Many mathematicians considered the fuzzy metric spaces in different view. First we recall the definition of continuous t-norm, fuzzy metric spaces and Cauchy sequences introduced by George and Veermani. This paper introduced some theorems related to this concept as fuzzy convergence and fuzzy continuity.

Definition 1.1

A binary operation * : [0, 1] × [0, 1] → [0, 1] is called a t-norm if ( [0, 1], *) is an abelian topological monoid with unit 1 such that a * b ≤ c × d whenever a ≤ c and b ≤ d for a, b, c, d ∈ [0, 1].

Examples of t-norms are a * b = a b and a * b = min{a, b}.

Definition 1.2

The 3–tuple (X, M, *) is said to be a fuzzy metric space if X is an arbitrary set * is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for all x, y, z ∈ X and t, s > 0

(i) $M(x, y, 0) > 0$,
(ii) $M(x, y, t) = 1$ for all t > 0 if and only if x = y,
(iii) $M(x, y, t) = M(y, x, t)$.
(iv) \( M(x, y, t) \circ M(y, z, s) \leq M(x, z, t + s) \) for all \( t, s > 0 \).
(v) \( M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1] \) is continuous.

**Definition 1.3**

A sequence \( \{x_n\} \) in a fuzzy metric space \((X, M, \ast)\) is a Cauchy sequence if and only if for each

\[ 0 < \varepsilon < 1 \text{ and } t > 0 \text{ there exist } n_0 \in \mathbb{N} \text{ such that for all } n, m \geq n_0 \text{ we have} \]

\[ M(x_n, x_m, t) > 1 - \varepsilon. \]

A fuzzy metric space is said to be **complete** if and only if every Cauchy sequence is convergent.

**Definition 1.4**

The 3–tuple \( (X, N, \ast) \) is said to be a **fuzzy normed space** if \( X \) is a vector space \( \ast \) is a continuous \( t \)-norm and \( N \) is a fuzzy set on \( X \times (0, \infty) \) satisfying the following conditions for every \( x, y \in X \) and \( t, s > 0 \):

(i) \( M(x, t) > 0 \),
(ii) \( M(x, t) = 1 \) if and only if \( x = 0 \),
(iii) \( M(\alpha x, t) = N(x, t/|\alpha|) \), for all \( \alpha = 0 \),
(iv) \( M(x, t) \ast M(y, s) \leq M(x+y, t+s) \) for all \( t, s > 0 \),
(v) \( M(x, \cdot) : (0, \infty) \rightarrow [0, 1] \) is continuous
(vi) \( \lim_{t \to 0^+} N(x, t) = 1 \).

**Definition 1.5**

Let \((X, N, \ast)\) be a fuzzy normed space. We define the open ball \( B(x, r, t) \) and the closed ball \( B[x, r, t] \) with center \( x \in X \) and radius \( 0 < r < t, t > 0 \), as follows

\[ B(x, r, t) = \{ y \in X : N(x-y, t) > 1-r \} \]

\[ B[x, r, t] = \{ y \in X : N(x-y, t) \geq 1-r \}. \]

**Definition 1.6**

A fuzzy metric space \((X, M, \ast)\) is called **compact** if every sequence has a convergent subsequence.

**Definition 1.7**

The fuzzy normed space \((X, N, \ast)\) is said to be a **fuzzy Banach space** whenever \( X \) is complete with respect to the fuzzy metric induced by fuzzy norm.

**Fuzzy Norm Spaces**
Lemma 2.1

Let \( N \) be a fuzzy norm. Then

(i) \( N(x, t) \) is non-decreasing with respect to \( t \) for each \( x \in X \).

(ii) \( N(x - y, t) = N(y - x, t) \).

**Proof:**

Let \( t < s \). Then \( k = s - t > 0 \) and we have

\[
N(x, t) = N(x, t) * 1
= N(x, t) * N(0, k)
\leq N(x, s).
\]

This proves the (i). To prove (ii) we have

\[
N(x - y, t) = N((-1)(y - x), t)
= N(y - x, \frac{t}{|t|})
= N(y - x, t).
\]

Lemma 2.2

A fuzzy metric \( M \) which is induced by a fuzzy norm on a fuzzy normed space \((X, N, *)\) has the following properties for all \( x, y, z \in X \) and every scalar \( \alpha \neq 0 \):

(i) \( M(x + z, y + z, t) = M(x, y, t) \),

(ii) \( M(\alpha x, \alpha y, t) = M(x, y, \frac{t}{|\alpha|}) \).

**Proof**

\[
M(x + z, y + z, t) = N((x + z) - (y + z), t)
= N(x - y, t) = M(x, y, t)
\]

Also,

\[
M(\alpha x, \alpha y, t) = N(\alpha x - \alpha y, t)
= N(x - y, \frac{t}{|\alpha|})
= M(x, y, \frac{t}{|\alpha|})
\]

**Example**

Let \((x, k, k)\) be a normed space. We define \( a * b = a \) or \( a * b = \min(a, b) \) and \( N(x, t) = \frac{k t^n}{k t^{n+m} + m ||x||} \)

Then \((X, N, *)\) is a fuzzy normed space. In particular if \( k = n = m = 1 \) we have \( N(x, t) = \frac{t}{t + ||x||} \) which is called the standard fuzzy norm induced by norm \( ||.|| \).
Theorem 2.3

Let \((X, M)\) be a fuzzy metric space with \(a \ast b = \text{Min} (a, b)\).

Let \(f_i : X \to X\) be a function with at least one fixed point \(x_i\) for each \(i = 1, 2, \cdots\), and \(f_0 : X \to X\) be a fuzzy contraction mapping with fixed point \(x_0\). If the sequence \((f_i)\) converges uniformly to \(f_0\), then the sequence \((x_i)\) converges to \(x_0\).

Proof:

Let \(k \in (0, 1)\) and choose a positive number \(N \in \mathbb{N}\) such that \(i \geq N\) implies \(M (f_i x f_0 x_i, (1-k) t) > 1 - r\) where \(r \in (0, 1)\) and \(x \in X\). Then, if \(i \geq N\),

we have 
\[
M(x_i, x_0, t) = M(f_i x f_0 x_i, t) \\
\geq M (f_i x f_0 x_i, (1-k) t) \ast M(f_i x f_0 x_i, k t) \\
> \text{Min} (1-r, M(x_i, x_0, t)).
\]

Hence, \(M(x_i, x_0, t) \to 1\) as \(i \to \infty\).

This proves that \((x_i)\) converges to \(x_0\). In what follows:

\(\pi_1 : X \times Y \to X\) will denote the first projection mapping defined by

\(\pi_1 (x, y) = x\), while \(\pi_2 : X \times Y \to Y\) will denote the second projection mapping defined by \(\pi_2 (x, y) = y\).

Lemma 2.4

If \((X, N, \ast)\) is a fuzzy normed space, then

(a) The function \((x, y) \to x + y\) is continuous,

(b) The function \((\alpha, x) \to \alpha x\) is continuous.

Proof:

If \(x_n \to x\) and \(y_n \to y\), then as \(n \to \infty\),

\[
N ((x_n + y_n) - (x + y), t) \geq N x_n - x, \frac{t}{2} \ast N y_n - y, \frac{t}{2} \to 1.
\]

This proves (a).

Now if \(x_n \to x\), \(\alpha_n \to \alpha\) and \(\alpha_n \neq 0\) then

\[
N (\alpha_n x_n - \alpha x, t) = N(\alpha_n (x_n - x) + x(\alpha_n - \alpha), t) \\
\geq N \frac{x_n - x}{\alpha_n} \ast N x (\alpha_n - \alpha), \frac{t}{2} \\
= N x_n - x, \frac{t}{\alpha_n} \ast N x, \frac{t}{\alpha_n - \alpha} \to 1,
\]

as \(n \to \infty\), and this proves (b).
Theorem 2.5

If \( M \) is a closed subspace of fuzzy normed space \( X \) and \( N(x+ M , t) \) is defined as above then

(a) \( N \) is a fuzzy norm on \( X/M \).

(b) \( N (Q x, t) \geq N(x , t) \).

(c) If \( (X , N, *) \) is a fuzzy Banach space, then so is \( (X /M,N,*) \).

Proof:

It is clear that \( N(x + M , t) \geq 0 \). Let \( N(x + M , t) = 1 \).

By definition there is a sequence \( \{x_n\} \) in \( M \) such that \( N(x + x_n , t) \rightarrow 1 \).

So \( x + x_n \rightarrow 0 \) or equivalently \( x_n \rightarrow (−x) \) and since \( M \) is closed so \( x \in M \) and \( x + M = M \), the zero element of \( X /M \). On the other hand we have,

\[
N ((x + M)+(y + M), t) \geq N((x + m)+(y + m), t) \geq N(x + m, t) \cdot N(y + m, t_2)
\]

for \( m, n \in M, x, y \in X \) and \( t_1 + t_2 = t \).

Now if we take sup on both sides, we have,

\[
N ((x + M)+(y + M), t) \geq N(x + M, t_1) \cdot N(y + M, t_2).
\]

Also we have,

\[
N (\alpha (x + M) , t)=N(\alpha x + M , t)
\]

\[
= sup \{N(\alpha x + \alpha y, t) : y \in M\}
\]

\[
= sup \{ N(\alpha x + \frac{y}{\alpha} , t) : y \in M\}
\]

\[
= N(x + M , \frac{t}{|\alpha|})
\]

Therefore \( (X, N, *) \) is a fuzzy normed space.

To prove (b) we have,

\[
N (Q x, t)= N(x + M , t)
\]

\[
= sup \{N(x + y, t) : y \in M\}
\]

\[
\geq N(x , t).
\]

Let \( \{x_n + M\} \) be a Cauchy sequence in \( X/M \). Then there exists \( \epsilon_n > 0 \) such that \( \epsilon_n \rightarrow 0 \) and,

\[
N ((x_n + M) −(x_{n+1} + M) , t) \geq 1 − \epsilon_n.
\]

Let \( y_1 = 0 \). We choose \( y_2 \in M \) such that,

\[
N(x_1 − (x_2 − y_2), t) \geq N((x_1 − x_2) + M, t) \cdot (1 − \epsilon_1).
\]
But \( N((x_1 - x_2) + M, t) \geq (1 - \varepsilon_1) \).

Therefore,\[ \quad N(x_1 - (x_2 - y_2), t) \geq (1 - \varepsilon_1)(1 - \varepsilon_1). \]

Now suppose \( y_{n-1} \) has been chosen, \( y_n \in M \) can be chosen such that\[ \quad N((x_{n-1} + y_{n-1}) - (x_n + y_n), t) \geq N((x_{n-1} - (x_n) + M, t) \ast (1 - \varepsilon_n - 1), \]

and therefore,\[ \quad N((x_{n-1} + y_{n-1}) - (x_n + y_n), t) \geq (1 - \varepsilon_n - 1) \ast (1 - \varepsilon_n - 1). \]

Thus, \( \{ x_n + y_n \} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, there is an \( x_0 \) in \( X \) such that \( x_n + y_n \rightarrow x_0 \) in \( X \). On the other hand \[ x_n + M = Q(x_n + y_n) \rightarrow Q(x_0) = x_0 + M. \]

Therefore every Cauchy sequence \( \{ x_n + M \} \) is convergent in \( X/M \) and so \( X/M \) is complete and \( (X/M, N, \ast) \) is a fuzzy Banach space.

**Theorem 2.6.**

Let \((X_1, M_1, \ast)\) and \((X_2, X_2, \ast)\) be fuzzy metric spaces under the same continuous \( \ast \)-norm \( \ast \). Let \( U \) denote the neighborhood system in \((X_1 \times X_2, M_1, \ast)\) and let \( V \) denote the neighborhood system in \((X_1 \times X_2, M_2, \ast)\) consisting of the Cartesian products \( B_{1} r, t) \times B_{2} r, t) \) where \( x_1 \in X_1, x_2 \in X_1, r \in (0,1) \) and \( t > 0 \). Then \( U \) and \( V \) induce the same fuzzy topology on \((X_1 \times X_2, M_1, \ast)\).

**Proof:**

Clearly, since \( \ast \) is continuous, \( U \) and \( V \) are bases for their respective topology. So, it is suffices to prove that for each \( V' \in V \) there exists a \( U' \in U \) such that \( U' \subseteq V' \), and conversely.

Let \( A_1 \times A_2 \in V \).

Then there exist neighborhoods \( B_{1} r, t) \) and \( B_{2} r, t) \) contained in \( A_1 \) and \( A_2 \) respectively.

Let \( r = \text{Min}(r_1, r_2), t = \text{Min}(t_1, t_2) \), and let \( x = (x_1, x_2) \). Here, we shall show that \( B_{x} r, t) \subseteq A_1 \times A_2 \).

Let \( y = (y_1, y_2) \in B_{x} r, t) \),

then we have \( M_1(x_1, y_1, t_1) \geq M_1(x_1, y_1, t_1) \ast 1 \)

\[ \geq M_1(x_1, y_1, t_1) \ast M_1(x_2, y_2, t_2) \]

\[ \geq M_1(x_1, y_1, t_1) \ast M_1(x_2, y_2, t_1) = M(x, y, t) > 1 - r \]

\[ \geq 1 - r_1. \]

Similarly,

we can show that \( M_2(x_2, y_2, t_2) > 1 - r_2 \). Thus \( y_1 \in r_1 B_{x_1} r_1, t_1) \) and
Conversely, suppose that $B_x (r, t) \in U$.

Since $\ast$ is continuous, there exists an $\eta \in (0, 1)$ such that $(1-\eta) \ast (1-\eta) > 1-r$.

Let $y \in (y_1, y_2) \in B_{x_1} (\eta, t) \times B_{x_2} (\eta, t)$.

Then $M \ast (x, y, t) = M_1 (x_1, y_1, t) \ast M_2 (x_2, y_2, t)$

\[ \geq (1-\eta) \ast (1-\eta) > 1-r \]

so that $y \in B_x (r, t)$ and $B_{x_1} (r, t) \times B_{x_2} (r, t) \subseteq B_x (r, t)$.

This completes the proof.

Theorem 2.7.

Let $M$ be a closed subspace of a fuzzy normed space $(X, N, t)$. If a couple of the spaces $X, M, X/M$ are complete, so is the third one.

Proof:

If $X$ is a fuzzy Banach space, so are $X/M$ and $M$.

Therefore all that needs to be checked is that $X$ is complete whenever both $M$ and $X/M$ are complete.

Suppose $M$ and $X/M$ are fuzzy Banach spaces and let $(x_n)$ be a Cauchy sequence in $X$.

Since $N((x_n - x_m) + M, t) \geq N(x_n - x_m, t)$

whenever $m, n \in N$, the sequence $(x_n + M)$ is Cauchy in $X/M$ and so converges to $y + M$ for some $y \in M$. So there exists a sequence $(\epsilon_n)$ such that $\epsilon_n \to 0$ and

$N((x_n - y) + M, t) > 1 - \epsilon_n$ for each $t > 0$.

Now by last theorem there exists a sequence $(\epsilon_n)$ in $X$

such that $y_n + M = (x_n - y) + M$ and

$N(y_n, t) > N((y_n - y) + M, t) \ast (1 - \epsilon_n)$.

So

$\lim_n N(y_n, t) \geq 1$ and

$\lim_n y_n = 0$.

Therefore $(x_n - y_n - y)$ is a Cauchy sequence in $M$ and thus is convergent to a point $z \in M$ and this implies that $(x_n)$ converges to $z + y$ and $X$ is complete.

Theorem 2.8. (Closed graph theorem)

Let $T$ be a linear operator from the fuzzy Banach space $(X, N_1, \ast)$ into the fuzzy Banach space $(Y, N_2, \ast)$. Suppose for every sequence $(x_n)$ in $X$

such that $x_n \to x$ and $Tx_n \to y$ for some elements $x \in X$ and $y \in Y$

it follows $T x = y$. Then $T$ is continuous.
Proof:

At first it is proved that the fuzzy norm N which is defined on \( X \times Y \) by,

\[
N((x, y), t) = N_1(x, t) \ast N_2(y, t)
\]

is a complete fuzzy norm. For each \( x, z \in X \), \( y, u \in Y \) and \( t, s > 0 \) it follows:

\[
N((x, y), t) \ast N((z, u), s) = [N_1(x, t) \ast N_2(y, t)] \ast [N_1(z, s) \ast N_2(u, s)]
\]

\[
\leq N_1(x + z, t + s) \ast N_2(y + u, t + s)
\]

\[
= N((x + z, y + u), t + s).
\]

Now if \( \{ (x_n, y_n) \} \) is a Cauchy sequence in \( X \times Y \), then for ever \( \varepsilon > 0 \) and \( t > 0 \) there exists \( n_0 \in N \) such that for \( m, n > n_0 \),

\[
\varepsilon \ast N((x_n, y_n) - (x_m, y_m), t) > 1 - \varepsilon.
\]

So for \( m, n > n_0 \),

\[
N_1(x_n - x_m, t) \ast N_1(y_n - y_m, t) = N((x_n - x_m, y_n - y_m), t)
\]

\[
= N((x_n, y_n) - (x_m, y_m), t)
\]

\[
> 1 - \varepsilon.
\]

Therefore \( \{ x_n \} \) and \( \{ y_n \} \) are Cauchy sequences in \( X \) and \( Y \) respectively and there exist \( x \in X \) and \( y \in Y \) such that \( x_n \to x \) and \( y_n \to y \) and consequently \( (x_n, y_n) \to (x, y) \). Hence \( (X \times Y, N, \ast) \) is a complete fuzzy normed space.

Theorem 2.9:

Every identity fuzzy function is a fuzzy continuous function in fuzzy normed space.

Proof:

For all \( \varepsilon \in (0, 1), t > 0 \) there exist \( \delta \in (0, 1), S = t > 0 \) such that, \( \varepsilon > \delta, x \in X: N(x_n - x, S) > 1 - \delta \)

\[
N(f(x_n) - f(x), t) = N(f(x_n - x), t)
\]

\[
= N((x_n - x), S) > 1 - \delta > 1 - \varepsilon
\]

Therefore \( f \) is a fuzzy continuous at \( x \in X \), since \( X \) is arbitrary point in \( X \).

References


