FUZZY NORMED AND BANACH SPACE IN FUZZYTOPOLOGICAL SPACES

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Abstract

The main aim of this paper is to consider the fuzzy normed spaces and define the fuzzy Banach spaces and also the introduction of fuzzy metric spaces of its quotients and prove some theorems and lemma's with the example. Based on the open mapping and closed graph theoremes on these fuzzy metric spaces.

Keyword

Fuzzy space , Fuzzy normed space, Fuzzy Banach space , Fuzzy metric space.

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1. Introduction

Many mathematicians have studied fuzzy normed spaces from several angles. The theory of fuzzy sets was introduced by L. Zadeh in 1965. The concept of fuzzy norm was introduced by Katsaras in 1984. Many mathematicians considered the fuzzy metric spaces in different view. First we recall the definition of continuous t-norm, fuzzy metric spaces and Cauchy sequences introduced by George and Veermani .This paper introduced some theorems related to this concept as fuzzy convergence and fuzzy continuity.

Definition 1.1

A binary operation $*: [0, 1] \times [0, 1] \to [0, 1]$ is called a **t-norm** if ([0, 1], *) is an abelian topological monoid with unit 1 such that $a * b \leq c \times d$ whenever $a \leq c$ and $b \leq d$ for $a, b, c, d \in [0, 1]$.

Examples of t-norms are $a * b = a\, b$ and $a * b = \min\{a, b\}$.

Definition 1.2

The 3 –tuple (X, M, $*$) is said to be a **fuzzy metric space** if X is an arbitrary set $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and

$t, s > 0$

(i) $M(x,y,0) > 0$,
(ii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,

(iv) M(x, y, t) $*$ M (y, z, s)$\leq$ M(x, z, t + s) for all t, s $>$ 0,
(v) M(x, y, $\cdot$) : (0, ∞) $\rightarrow$ [0, 1] is continuous.

Definition 1.3

A sequence $\{x_n\}$ in a fuzzy metric space (X, M, $*$) is a Cauchy sequence if and only if for each

0< $\epsilon < 1$ and t $>$ 0 there exist $n_0 \in$ N such that for all n, m $\geq$ n_0 we have

M (x_n , x_m,t) $>$ 1- ϵ.

A fuzzy metric space is said to be **complete** if and only if every Cauchy sequence is convergent.

Definition 1.4

The 3 – tuple (X, N, $*$) is said to be a **fuzzy normed space** if X is a vector space $*$ is a continuous t-norm and N is a fuzzy set on

X $\times$ (0, ∞) satisfying the following conditions for every x, y $\in$ X and

t, s$>$ 0:

(i) M (x, t) $>$ 0,
(ii) M(x, t) =1 if and only if x =0,
(iii) M(αx, t) = N (x, t/ $|\alpha|$), for all α =0,
(iv) M(x, t) $*$ M (y, s)$\leq$ M(x,+y, t + s) for all t, s $>$ 0,

(v) M(x, $\cdot$) : (0, ∞) $\rightarrow$ [0, 1] is continuous

(vi) $lim_{t\to\infty}$ N(x, t) =1.

Definition 1 .5

Let (X , N, $*$) be a fuzzy normed space. We define the open ball

B(x, r, t) and the closed ball B[x, r, t] with center x $\in$ X and radius

0< $r <$ t, t $>$ 0, as follows

B(x, r, t) ={y $\in$ X : N (x-y, t) $>$ 1-r }

B[x, r, t] ={y $\in$ X : N (x-y, t) $\geq$ 1-r }.

Definition 1 .6

A fuzzy metric space (X, M, $*$) is called **compact** if every sequence has a convergent subsequence.

Definition 1 .7

The fuzzy normed space (X , N, $*$) is said to be a **fuzzy Banach space** whenever X is complete with respect to the fuzzy metric induced by fuzzy norm.

Fuzzy Norm Spaces

Lemma 2.1

Let N be a fuzzy norm. Then

(i) $N(x ,t)$ is non decreasing with respect to t for each $x \in X$.

(ii) $N(x - y ,t)=N(y - x ,t)$.

Proof:

Let $t<s$. Then $k = s - t>0$ and we have

$$N(x, t) = N(x, t) * 1$$

$$= N(x, t) * N(0,k)$$

$$\leq N(x, s).$$

This proves the (i). To prove (ii) we have

$$N(x - y, t) = N((-1)(y - x),t)$$

$$= N\left(y - x , \frac{t}{|-1|}\right)$$

$$= N(y - x, t).$$

Lemma 2.2

A fuzzy metric M which is induced by a fuzzy norm on a fuzzy normed space $(X,N,*)$ has the following properties for all $x,y,z \in X$ and every scalar $\alpha \neq 0$:

(i) $M(x + z , y + z, t)=M(x, y, t)$,

(ii) $M(\alpha x , \alpha y ,t)=M\left(x, y, \frac{t}{|\alpha|}\right)$

Proof

$$M (x + z ,y + z, t)=N((x + z) - (y + z), t)$$

$$= N(x - y, t)=M(x ,y, t)$$

Also, $$M (\alpha x, \alpha y, t)=N(\alpha x - \alpha y, t)$$

$$= N\left(x - y, \frac{t}{|\alpha|}\right)$$

$$= M\left(x, y, \frac{t}{|\alpha|}\right)$$

Example

Let (x, k. k) be a normed space. We define $a * b = a b$ or $a * b = \min(a ,b)$ and $N(x, t) = \frac{kt^n}{kt^n+m||x||}$

Then $(X, N, *)$ is a fuzzy normed space. In particular if $k = n = m = 1$ we have $N (x, t)= \frac{t}{t+||x||}$ which is called the standard fuzzy norm induced by norm $||.||$

Theorem 2.3

Let (X, M $*$) be a fuzzy metric space with a $*$ b = Min (a ,b).

Let f_i: X → X be a function with at least one fixed point x_i for each i = 1,2,⋯, and f_o: X → X be a fuzzy contraction mapping with fixed point x0. If the sequence (f_i) converges uniformly to f_0, then the sequence (x_i) converges to x_0.

Proof:

Let k ∈ (0, 1) and choose a positive number N ∈ ℕ such that i ≥ N implies M ($f_i x, f_0 x$,(1−k) t) > 1− r where r ∈ (0,1) and x ∈ X. Then, if i ≥ N,

we have M(x_i, x_0,t)=M($f_i x, f_0 x$,t)

≥ M ($f_i x, f_0 x$,(1−k) t) $*$M($f_i x, f_0 x$,k t)

> Min (1− r, M (x_i, x_0, t)).

Hence, M (x_i, x_0, t) → 1 as i →∞.

This proves that (x_i) converges to x_0. In what follows

π_1 : X ×Y → X will denote the first projection mapping defined by

π_1 (x, y)=x, while π_2: X×Y → Y will denote the second projection mapping defined by π_2 (x, y)=y.

Lemma 2.4

If (X, N,$*$) is a fuzzy normed space, then

(a) The function (x, y) → x + y is continuous,

(b) The function (α , x) → αx is continuous.

Proof:

If x_n → x and y_n → y, then as n → ∞ ,

N ((x_n +y_n)−(x + y), t) ≥ N x_n− x, $\frac{t}{2}$ $*$ N y_n − y, $\frac{t}{2}$ → 1.

This proves (a).

Now if x_n → x, α_n → α and α_n ≠0 then

N ($\alpha_n x_n - \alpha x$, t)= N(α_n (x_n− x)+x(α_n− α), t)

≥ N α_n (x_n − x), $\frac{t}{2}$$*$ N x (α_n − α), $\frac{t}{2}$

= N x_n − x, $\frac{t}{2\alpha_n}$$*$ N x, $\frac{t}{2(\alpha_n-\alpha)}$ → 1,

as n → ∞ , and this proves (b).

Theorem 2.5

If M is a closed subspace of fuzzy normed space X and N(x+ M ,t) is defined as above then

(a) N is a fuzzy norm on X/M.

(b) N (Q x, t) ≥ N(x ,t).

(c) If (X , N,∗) is a fuzzy Banach space, then so is (X /M,N,∗).

Proof:

It is clear that N(x + M , t) ≥ 0. Let N(x + M, t) = 1.

By definition there is a sequence $\{x_n\}$ in M such that N(x +x_n , t) → 1.

So x + x_n → 0 or

equivalently x_n→ (−x) and since M is closed so x ∈ M and x + M = M, the zero element of X /M. On the other hand we have,

$$N((x + M)+(y + M), t)=N((x + y)+M, t)$$
$$\geq N((x + m)+(y + m),t)$$
$$\geq N(x + m,t_1) * N(y + m, t_2)$$

for m, n ∈ M, x ,y ∈ X and $t_1 + t_2 = t$.

Now if we take sup on both sides, we have,

$$N((x + M)+(y + M), t) \geq N(x + M, t_1) * N(y + M, t_2).$$

Also we have,

$$N(\alpha (x + M) ,t)=N(\alpha x + M ,t)$$
$$= \sup \{N(\alpha x + \alpha y, t) : y \in M\}$$
$$= \sup \{ N (x + y, \tfrac{t}{|\alpha|}) : y \in M\}$$
$$= N(x + M , \tfrac{t}{|\alpha|})$$

Therefore (X, N, ∗) is a fuzzy normed space.

To prove (b) we have,

$$N (Q x, t)= N(x + M ,t)$$
$$= \sup \{N(x + y, t) : y \in M\}$$
$$\geq N(x , t).$$

Let $\{x_n + M\}$ be a Cauchy sequence in X/M. Then there exists $\epsilon_n > 0$ such that $\epsilon_n \to 0$ and, $N((x_n + M) - (x_{n+1} + M) ,t) \geq 1 - \epsilon_n$.

Let $y_1 = 0$. We choose $y_2 \in$ M such that,

$$N(x_1 - (x_2 - y_2),t) \geq N((x_1 - x_2)+M, t) * (1 - \epsilon_1).$$

Bu t N ((x_1− x_2) +M ,t) ≥ (1 − ϵ_1).

Therefore, N(x_1 − (x_2 − y_2),t) ≥ (1 − ϵ_1)(1 − ϵ_1).

Now suppose y_{n-1} has been chosen, y_n∈ M can be chosen such that

N ((x_{n-1} + y_{n-1}) − (x_n+ y_n) ,t) ≥ N((x_{n-1} −(x_n)+M, t) ∗ (1 −ϵ_n −1),

and therefore,

N ((x_{n-1} + y_{n-1}) − (x_n+ y_n), t) ≥ (1 − ϵ_n −1) ∗ (1 − ϵ_n−1).

Thus, { x_n + y_n} is a Cauchy sequence in X. Since X is complete, there is an x_0 in X such that x_n + y_n → x_0 in X. On the other hand

x_n + M = Q (x_n +y_n) → Q(x_0)= x_0 + M.

Therefore every Cauchy sequence {x_n + M} is convergent in X/M and so X/M is complete and (X/M, N ,∗) is a fuzzy Banach space.

Theorem 2.6.

Let (X_1, M_1,∗) and (X_2, X_2,∗) be fuzzy metric spaces un- der the same continuous t-norm ∗. Let U denote the neighborhood system in ($X_1 \times X_2, M_*$,∗) and let V denote the neighborhood system in ($X_1 \times X_2$, M_*,∗) consisting of the Cartesian products B_x1 (r, t)× B_x2 (r, t) where $x_1 \in X_1$, x_2∈ X_1, r ∈ (0,1) and t >0. Then U and V induce the same fuzzy topology on ($X_1 \times X_2$, M_*,∗) .

Proof:

Clearly, since ∗ is continuous, U and V are bases for their respective topology. So, it is suffices to prove that for each V' ∈ V there exists a U' ∈ U such that $U' \subseteq V'$, and

conversely.

Let A_1 × A_2 ∈ V .

Then there exist neighborhoods B_{x1} (r, t) and B_{x2} (r, t) contained in A_1 and A_2 respectively.

Let r = Min(r_1, r_2), t = Min(t_1, t_2), and

let x =(x_1, x_2). Here, we shall show that B_x (r, t)∈ A_1 × A_2.

Let y =(y_1 × y_2) ∈ B_x (r, t),

then we have $M_1(x_1, y_1, t_1)$= $M_1(x_1, y_1, t_1)$∗1

≥ $M_1(x_1, y_1, t_1)$∗ $M_1(x_2, y_2, t_2)$

≥ $M_1(x_1, y_1, t_1$, t) ∗$M_1(x_2, y_2$,t)

= M(x ,y ,t) > 1−r

≥ 1− r_1.

Similarly,

we can show that $M_2(x_2, y_2, t_2)$ > 1−r_2.. Thus y_1∈ r_1 B_{x1} (r_1, t_1)and

$y_2 \in B_{x2}$ (r_2, t_2) which implies that B_x (r, t) ∈ $A_1 \times A_2$.

Conversely, suppose that B_x (r, t) ∈ U.

Since ∗ is continuous, there exists an η ∈ (0, 1) such that (1−η) ∗ (1−η) > 1−r.

Let y ∈ (y1y_1, y_2)∈ B_{x1} (η, t)× B_{x2} (η, t).

Then M∗(x, y, t)= $M_1(x_1, y_1, , \text{t})* M_2(x_2, y_2, \text{t})$

≥ (1−η) ∗ (1−η)

> 1−r

so that y ∈ B_x (r, t) and B_{x1} (r, t)× B_{x2} (r, t) ⊆ B_x (r ,t).

This completes the proof.

Theorem 2.7.

Let M be a closed subspace of a fuzzy normed space (X, N, t). If a couple of the spaces X, M, X/M are complete, so is the third one.

Proof:

If X is a fuzzy Banach space, so are X/M and M.
Therefore all that needs to be checked is that X is complete whenever both M and X/M are complete.

Suppose M and X/M are fuzzy Banach spaces and let $\{x_n\}$ be a Cauchy sequence in X.

Since N(($x_n - x_m$)+ M, t) ≥ N($x_n - x_m$, t)

whenever m ,n ∈ N, the sequence $\{x_n$ +M$\}$ is Cauchy in X/M and so converges to y + M for some y ∈ M.So there exists a sequence $\{\epsilon_n\}$ such that $\epsilon_n \to 0$ and

N ((x_n − y) + M ,t) > 1 − ϵ_n for each t>0.

Now by last theorem there exists a sequence $\{\epsilon_n\}$ in X

such that y_n + M = (x_n − y) + M and

N (y_n, t) >N((y_n − y) + M, t) ∗ (1 − ϵ_n).

So lim_n N(y_n, t) ≥ 1 and

lim_n y_n= 0.

Therefore $\{x_n - y_n$ – y $\}$ is a Cauchy sequence in M and thus is convergent to a point z ∈ M and this implies that $\{x_n\}$ converges to z + y and X is complete.

Theorem 2.8. (Closed graph theorem)

Let T be a linear operator from the fuzzy Banach space (X,N_1, ∗) into the fuzzy Banach space (Y, N_2, ∗). Suppose for every sequence $\{x_n\}$ in X

such that x_n → x and Tx_n → y for some elements x ∈ X and y ∈ Y

it follows T_x= y. Then T is continuous.

Proof:

At first it is proved that the fuzzy norm N which is defined on X ×Y by,

$$N((x, y),t)= N_1(x ,t) * N_2(y, t)$$

is a complete fuzzy norm. For each x ,z ∈ X , y, u ∈ Y and t, s > 0 it follows:

$$\begin{aligned} N((x ,y), t) * N((z ,u), s) &= [N_1(x, t) * N_2(y, t)] * [N_1(z, s) * N_2(u, s)] \\ &= [N_1(x, t) * N_1(z, s)] * [N_2(y, t) * N_2(u, s)] \\ &\le N_1(x + z, t+ s) * N_2(y + u, t + s) \\ &= N((x + z, y + u), t+ s). \end{aligned}$$

Now if $\{(x_n, y_n)\}$ is a Cauchy sequence in X × Y , then for ever ε >0 and t >0 there exists $n_0 \in$ N such that for

m, n > n_0,

$$\varepsilon\ N((x_n, y_n) - (x_m, y_m), t) > 1 - \varepsilon .$$

So for m, n > n_0 ,

$$\begin{aligned} N_1(x_n - x_m, t) * N_1(y_n - y_m, t) &= N((x_n - x_m , y_n - y_m), t) \\ &= N((x_n, y_n) - (x_m, y_m) ,t) \\ &> 1 - \varepsilon. \end{aligned}$$

Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and Y respectively and there exist x ∈ X and y ∈ Y such that $x_n \to$ x and $y_n \to$ y and consequently $(x_n, y_n) \to$ (x, y). Hence (X × Y, N, $*$) is a complete fuzzy normed space.

Theorem 2.9:

Every identity fuzzy function is a fuzzy continuous function in fuzzy normed space.

Proof:

For all $\varepsilon \in (0, 1)$, t >0 there exist $\delta \in (0, 1)$, S= t >0

such that , $\varepsilon > \delta$,x ∈ X: N $(x_n$ –x ,S) $> 1- \delta$

$$\begin{aligned} N(f(x_n) - f(x) ,t) &= N(f(x_n - x) , t) \\ &= N(x_n - x) , s) > 1 - \delta > 1 - \varepsilon \end{aligned}$$

Therefore f is a fuzzy continuous at x ∈ X, since X is arbitrary point in x.

then f is a fuzzy continuous function.

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