FUZZY UNIFORM CONTINUITY THEOREM ALONG WITH FUZZY CONTINUOUS IN A FUZZY NORMED LINEAR SPACE

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ABSRACT

The aim of this paper is introduce the concept of uniform continuity theorem along with the variable τ and μ and also introduce the banach contradiction principle of fuzzy linear operators in a fuzzy normed linear space based on our results.

Keyword: *Fuzzy uniform continuity; fuzzy normed linear space, fuzzy linear operators.*

1 INTRODUCTION AND PRELIMINARIES:

The concept of The fuzzy set introduced by L.Zadeh [14] in 1965. If X is a nonempty set a fuzzy set in X is a function μ from X into the unit interval [0,1]. The classical union and intersection of ordinary subsets of X can be extended by the following formulas, proposed by L.Zadeh

$$(\vee_{i \subset I} \mu_i)\ (x) = \sup\{\mu_i\ (x) : i \in I\}, \quad (\wedge_{i \subset I} \mu_i)\ (x) = \inf\{(x)\ i \in I\ \}.$$

From here to the notion of fuzzy topological space.there was one more step to be taken. Thus in 1968, C.L. Chang [4] introduced the notion of fuzzy topological space. The definition is a natural translation to fuzzy sets of the ordinary definition of topological space. Indeed, a fuzzy topology is a family $\mathcal{T}$, of fuzzy sets in X, such that $\mathcal{T}$ is closed with respect to arbitrary union and finite intersection and every constant function belong to $\mathcal{T}$.

One of the important problems concerning the fuzzy topological spaces is to obtain an adequate notion of fuzzy metric space. Many authors have investigated this question and several notions of fuzzy metric space have been defined and studied. We just mention the definition given by I. Kramosil and J. Michalek [9] in 1975.

1.1 Definition

The pair (X,M) is said to be a fuzzy metric space if X is an arbitrary set and M is a fuzzy set in X × X × [0,∞) satisfying the following conditions:

(M1) $M(x, y, 0) = 0$, $(\forall)x$, y ∈ X;

(M2) $(\forall)x$,y ∈ X, x = y if and only if $M(x, y, t) = 1$ for all t> 0;

(M3) $M(x, y, t) = M(y, x, t)$,$(\forall)x$,y ∈ X, (∀)t>0;

(M4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$, $(\forall)x$, y, z ∈ X, (∀)t, s > 0;

(M5) $(\forall)x$, y ∈ X, $M(x, y, .)$: [0, ∞) → [0, 1] is left continuous and $\lim_{t\to\infty} M(x, y, t) = 1$.

We note that , in previous definition * denotes a continuous t-norm (see [13]). The basic examples of continuous t-norm are ⋀, ., *L, which are defined by a ⋀ b = $min\{a, b\}$,a .b = ab (usual multiplication in [0, 1]) and a* L^b = max { $a + b - 1$, 0 } (the Lukasiewicz t-norm) .

In studying fuzzy topological linear spaces , A. K. Katasaras [8], in 1984, first introduced the notion of fuzzy norm on a linear space. Since then many mathematicians have introduced several notions of fuzzy norm from different points of view. Thus, C. Felbin [6] in 1992 introduced an idea of fuzzy norm on a linear space by assigning a fuzzy real number to each element of linear space. In 1994, S.C. Cheng and J. N. Mordeson [5] introduced a concept of fuzzy norm on a linear space whose associated metric is Kramosil and Michalek type. Following S.C. Cheng and J.N. Mordeson, in 2003, T. Bag and S.K. Samanta [2] proposed another concept of fuzzy norm.

In this paper we continue the study of fuzzy continuous mappings in fuzzy normed linear spaces intiated by T. Bag and S.K. Samanta [3], as well as by 1. Sadeqi and F.S. Kia [12], in a more general settings:

1.2 Definition [10]

Let X be a vector space over a field K (where K is R or C) and * be a continuous t-norm. A fuzzy set N in X × [0, ∞) is called a fuzzy norm on X if it satisfies:

(N1) $N(x, 0) = 0$, $(\forall)x \in X$;

(N2) [$N(x, t) = 1$, (∀) t > 0] if and only if = 0;

(N3) $N(\lambda x, t) = N\left(x, \frac{t}{|\lambda|}\right)$, $(\forall)x \in X$, (∀) t ≥ 0, (∀) $\lambda \in K^*$;

(N4) $N(x + y, t + s) \geq N(x, t) * N(y, s)$, (∀) $x, y \in X$, (∀) $t, s \geq 0$;

(N5) $(\forall)x \in X, N(x, .)$ is left continuous and $\lim_{n\to\infty} N(x, t) = 1$.

The triple (X, N, *) will be called fuzzy normed linear space (briefly FNLS).

(Remark 3. a) T. Bag and S.K. Samanta [2], [3] gave a similar definition for * = ⋀, but in order to obtain some important results they assumed that the fuzzy norm also satisfied the following conditions:

(N6) $\mathrm{N}(x,t) > 0, (\forall)t > 0 \Rightarrow x = 0$;

(N7) $(\forall)x \neq 0$, $\mathrm{N}(x,.)$ is a continuous function and strictly increasing on the subset $\{t : 0 < N(x,t) < 1\}$ of R.

The results obtained by T. Bag and S.K. Samanta [3], as well as by I. Sadeqi and F.S. Kia [12], can be found in this more general settings.

b) I. Golet [7], C. Alegre and S. Romaguera [1] also gave this definition in the context of real vector spaces.

c) $\mathrm{N}(x,.)$ is nondecreasing, $(\forall)\ x \in X$.

1.3 Example [2] Let X be a linear space and ‖.‖ be a norm on X. let

$$N(x,t) := \begin{cases} 1\ if\ \ |x| < t \\ 0\ if\ \ |x| \geq t \end{cases}$$

Then (X,N,⋀) is a FNLS. In particular, (C,N, ⋀) is a FNLS.

1.4 Definition |2|

Let$(X,N,*)$ be a FNLS and (x_n) be a sequence in X,

1. The sequence (x_n) is said to be convergent if$(\exists)x \in X$ such that
$$\lim_{n\to\alpha} N\,(x_n - x, t) = 1.\,(\forall)t > 0.$$
In this case,x is called the limit of the sequence (xn) and we denote
$$\lim_{n\to\alpha} x_n = x\ or\ x_n \to x$$
2. The sequence (x_n) is called Cauchy sequence if
$$\lim_{n\to\alpha} N\left(x_{n+p} - x_n, t\right) = 1.\,(\forall)t > 0.\,(\forall)p \in N^*$$
3. (X,N,*) is said to be complete if any Cauchy sequence in X is convergent in a point in X. A complete FNLS will be called a fuzzy banach space.

1.5 Theorem Let (X,N,*) be a FNLS and $p_\propto(x) \coloneqq inf\{t > o : N(x,t) > \alpha\}, \alpha \in (0,1)$

Then, for $x \in X, s > 0, \alpha \in (0,1), we\ have:$

$$p_\alpha(x) < s\ if\ and\ only\ if\ N(x,s) > \alpha$$

Proof: The proof is entirely the same as in |10|,where there are considered FNLSs of type (X,N,A)

The structure of the paper is as follows: in section 2,we introduce the notion of uniformly fuzzy continuous mapping and we establish the uniform continuity theorem in fuzzy settings. The concept of fuzzy lipschitzian mapping is introduced and s fuzzy version for Banach's contraction principle is obtained. In section 3,special attention is given to various characterizations of fuzzy continuous linear operators. Based on our results, classical principles of functional analysis (such as the uniform boundedness principle the open mapping theorem and the closed graph theorem)can be extended in a more general fuzzy context.

Even if the structure of fuzzy F-spaces, recently introduced in |11|,is much more complicated than that of fuzzy banachspaces,we intent to study, in a further paper, fuzzy continuous linear operators on fuzzy F-spaces and to prove that the well-known principles of functional analysis are valid in this context too.

In the following sections (X,$N_1 *_1$),(Y,N_2,$*_2$) will be FNLSs with the t-norm $*_1$, $*_2$ which satisfy $sup_{x\in(0,1)} x *_i\ x$=1,for all i=1,2,3…..

2. FUZZY CONTINUOUS MAPPINGS

2.1 Definition . |3| A mapping T : X →Y is said to be fuzzy continuous at x0∈ X , if

(∀) $\boldsymbol{\varepsilon}$> 0, (∀)$\boldsymbol{\alpha}$ ∈ (0,1), (∃)$\boldsymbol{\delta} = \boldsymbol{\delta}(\boldsymbol{\varepsilon},\boldsymbol{\alpha},\boldsymbol{x0}) > \mathbf{0}$, (∃)$\boldsymbol{\beta} = \boldsymbol{\beta}(\boldsymbol{\varepsilon},\boldsymbol{\alpha},\boldsymbol{x0}) \in (\mathbf{0},\mathbf{1})$

Such that (∀)$x \in X: N1(x - x0.\delta) > \beta$ we have that $N2(T(x) - T(x0).\epsilon) > \alpha$

If T is fuzzy continuous at each point of X, then T is called fuzzy continuous on X.

2.2 Theorem |3| A mapping T: X→ Y is fuzzy continuous at $xo\ \in X, if\ for\ all$

(∀)(x_n)⊆ X, $x_n \to x_0, we\ have\ that\ T(x_n) \to T(xo)$.

Proof :

Given **:** A mapping $T: X \to Y$ is a fuzzy continuous at $x_o \in X$

Proof : $T(x_n) \to T(x_0)$

Let $\varepsilon > o\ and\ \alpha\ \in (0,1)$

T:$X \to y$ is a fuzzy continuous on X ∀

$$x_o \in X\ there\ exists\ \delta_x = \delta\frac{\varepsilon}{2}, \alpha_0, x > 0$$

$$\beta_x = \beta\frac{\varepsilon}{2}, \alpha_{0,}\ x\ \in (0,1)$$

Such that

$$(\forall) Y \in X,$$

$$N_1(x - y), \delta_x) > \beta_x$$

$$Then\ \ N_2\, T\,(x_n) - T(x_0)\,, \frac{\varepsilon}{2} > \alpha_0$$

$Let\ \beta = max(\gamma_x)\ and\ x = B_{i=1}^{s}\ x_i\ , \gamma_x\ ; \frac{\delta_x}{2}, \delta = min\frac{\delta_x}{2}\ for$ i=1,2,3,……

$$\delta = \min \frac{\delta_{xi}}{2} \quad for\ i = 1,2,3, \dots . n$$

Let $x_0, x_n\ \epsilon\ x_i$

Such that $N\ (x - y, \delta) > \beta, \quad x_0 \epsilon X$ there exists

$i\epsilon\{1,2,3, \dots . n\}$, $x_0 \epsilon \beta$

$$x_0\ \epsilon\ x_i, \gamma_x; \frac{\delta x_i}{2}$$

$$N_i\left(x_0 - x_n, \frac{\delta x_i}{2}\right) > \gamma_x\ ;$$

$$N_i(x_0 - x_n, \delta x_i) \geq N_i\ ;$$

$x_0 - x_n\ \delta x_i > \gamma_{xi}$;

$$> \beta_{xi}$$

$$N_2\ T(x_0) - T\ (x_n), \frac{\varepsilon}{2} > \alpha_0$$

$$N_i(x_0 - x_n, \delta x_i) \geq N_i\ ;$$

$$x_0 - x_n, \frac{\delta x_i}{2} *, N_i$$

$$x_0 - x_n, \frac{\delta x_i}{2} \geq\ N_i(x_0 - x_n, \delta) x_i$$

$N_i\left(x_0 - x_n, \frac{\delta x_i}{2}\right) > \beta *\ \gamma_{xi} \geq \gamma_{xi} * \gamma_x; > \beta_{xi};$

$$N_2\ T(x_0) - T\ (x_n), \frac{\varepsilon}{2} > \alpha_0$$

$N_2\ T(x_0) - T\ (x_n), \epsilon \geq N_2\ T(x_0) - T(x_n), \frac{\varepsilon}{2} *_2\ N_2\ T\ (x_n) - T(x_0), \frac{\varepsilon}{2} > \alpha_0 *_2\ \alpha_0 > \alpha.$

Hence $T(x_n) \to T(x_0)$.

2.3 Theorem (**uniform continuity theorem)** Let $(X, N_1, * 1)$ be a compact FNLS and $(Y, N_2, * 2)$ be a FNLS.

If $T: X \to Y$ is a fuzzy continuous mapping ,then T is uniformly fuzzy continuous.

Proof :Let $\varepsilon > 0$ *and* $\alpha \epsilon (0,1)$.

As $sup_{x\in(0,1)} x * 2\ x = 1$,then there exists $\alpha_0 \epsilon (0,1)$ such that $\alpha_0 *\ 2\alpha_{0>}\alpha$

As $T: X \to Y$ is a fuzzy continuous on X, for all $x \epsilon X$,then there exist $\delta_x = \delta \frac{\varepsilon}{2}(x_{0,x}) > 0 \beta_{x=}\beta \frac{\varepsilon}{2}, \alpha_0, x \epsilon (0,1)$

$(\forall) y \in X : N_1(x - y, \delta_x) > \beta_x \Rightarrow N_2\ T(y), \varepsilon/2 > \alpha_0$

As $sup_{x\in(0,1)} x * 1\ x = 1$,we can take $\gamma_x > \beta_x$

such that $\gamma_x * 1\ \gamma_x > \beta_x$

Since X is compact and $B\ x. \gamma_x . \frac{\delta_x}{2} x\epsilon\ X$ is an open covering of X,there exist $x_1, x_{2,..x_n}$

In X such that$X = \quad B\ x_i, \gamma_{xi}, \delta_{xi}/2$.

Let $\beta = max\{\gamma_{xi}\} and\ \delta = \min \delta_{\frac{xi}{2}},$ for i=1,2,3,…..n

Let $x, y\ \epsilon X$ arbitrary ,such that $N_1(x - y, \delta) > \beta$,

$$As\ x \epsilon\ X, there\ exists\ i\ \epsilon\ (1,2, \dots n)$$

Such that $x\epsilon\ B\ x_i, \gamma x_i. \delta x_{i/2}$, namely $N_1\ x - x_{i\ .}\ \delta x_{\frac{i}{2}} > \gamma_{xi}$. Hence

$$N_1(x - x_{i\ ,}, \delta_{xi}) \geq N_1\ \ x - x_{i\ .}\delta x_{\frac{i}{2}} > \gamma_{xi} > \beta_{xi}$$

$$N_2\ T(x) - T\ (x_i), \frac{\epsilon}{2} > \alpha_0$$

We remark that

$$N_1(y - x_{i\ ,}, \delta_{xi}) \geq N_1\ \ y - x_{i\ .}\delta x_{\frac{i}{2}} *_1 N_1\ \ x - x_{i\ .\delta_{xi/2}} \geq N_1(y - x, \delta) *_1 N_1\ x - x_{i\ .}\delta_{\frac{xi}{2}} > \ \beta * 1\ \gamma_{xi}$$
$$\geq 1\ \gamma_{xi} * 1\gamma_{xi} > \beta_{xi}$$

Thus $N_2\ T(y) - T(x_i). 2 > \alpha_0$.

In conclusion

$$N_2(T(x) - T(y), \varepsilon \geq N_2\ T(x) - T(x_i), \frac{\varepsilon}{2} * 2\ N_2 T(x_i) - T(y) \frac{\varepsilon}{2} > \alpha_0 * 2\alpha_0 > \alpha$$

2.4 Definition A mapping $T : X \to Y$ is said to be fuzzy Lipschitzian on X if$(\exists) L > 0$ such that

$$N_2(T(x) - T(y), t\ ) \geq\ N_1\ \ x - y, \frac{t}{L}\ \ . (\forall) t > 0. (\forall) x, y\ \in X$$

If L< 1 we say that T is a fuzzy contraction.

2.5 Remark It is clear that a fuzzy Lipschitzian mapping is necessarily fuzzy continuous.

2.6 Theorem (Banach's contraction principle). Let (X,N,*) be a fuzzy banach space and $T: X \to X$ be a fuzzy contraction.Then T has a unique fixed point z ∈ X and

$$\lim_{n\to\infty} T^n(x) = z. (\forall)x \in X$$

Proof :

Let $x \in X$ be arbitrary ,then $(T^n(x))$ is a Cauchy sequence .Indeed ,for t>0 and $p \in N^*$, we have

$$N(T^{n+p}(x) - T^n(x).t \geq N \quad T^{n+p-1}(x) - T^{n-1}(x), ?\frac{\delta}{L} \geq \cdots \dots \dots \dots \geq NT^p(x) - x, \frac{\delta}{L}$$

As $L \in (0,1), we\ that\ \lim_{n\to\infty} \frac{t}{L^n} = \infty, thus$

$$\lim_{n\to\infty} N\, T^p(x) - x. ?/L^n = 1$$

Hence $\lim_{n\to\infty} N(T^{n*p}(x) - T^n(x), \delta)$ 1,namely $(T^n(x))$ is a Cauchy sequence.

Since X is complete, we have that $(T^n(x))$ is a convergence sequence.

Thus $(\exists)z \in X\ such\ that\ \lim_{n\to\infty} T^n(x) = z.$ We note that

$$z = \lim_{n\to\infty} T^{n+1}(x) = \lim_{n\to\infty} T(T^n(x)) = \mathrm{T(z)}.$$

Now we show the uniqueness. Suppose that there exist $z, y \in X, z \neq y$ with the property

$z = T(z), y = T(y).$ As $z \neq y, there\ exists\ \delta > 0$ such that $N(z - y, \delta) = \alpha < 1.$ Then , for all $n \in N$,we have

$$\alpha = N(y - z, \delta) = N(T^n(y) - T^n(z), \delta) \geq N\ \ y - \frac{z\delta}{L^n} \to 1$$

Thus $\alpha = 1, which$ contradicts our assumption.

3. FUZZY CONTINUOUS LINEAR OPERATORS

3.1 Theorem Let T : X → Y be a linear operator. Then T is fuzzy continuous on X, if and only if T is fuzzy continuous at a point $x_0 \in X.$

proof : "$\Longrightarrow$ " *it is obvious*.

"$\Longleftarrow$ " let $y \in Y$ be arbitrary. We will show that T is fuzzy continuous at y. let $\varepsilon > 0, \alpha \in (0,1)$ since T is fuzzy continuous at $x_0 \in X. \delta > 0, \beta \in (0,1)$ such that

For all $x \in X$:$N_1(x - x_{0,}\delta)$>$\beta \Longrightarrow \boldsymbol{N_2}(T(x) - T(x_0), \varepsilon) > \alpha,$

Replacing $x\ by\ x + x_{0,} - y$, we obtain that

For all $x \in X$: $N_1(x + x_{0,} - y - x_{0,}\delta > x_{0,}\delta$> $\beta \Longrightarrow \boldsymbol{N_2}(\boldsymbol{T(x)} - \mathrm{T(y)}, \varepsilon > \alpha.$

Thus T is fuzzy continuous at $y \in Y$.As y is arbitrary, it follows that T is fuzzy continous on D(T).

3.2 Corollary A linear functional f: $(X, N_1,*) \rightarrow (C, N,\wedge)$is fuzzy continuous iff $(\exists)\beta \in(0,1)$, $(\exists)M > 0$ such that

For all t>o, for all $x \in X, N_1(x,t) > \beta \Rightarrow |f(x)| < Mt$.

Proof:

According to the previous theorem f is fuzzy continuous iff

$(\forall)\alpha \in (0,1)$, $(\exists)$ $\beta \in(0,1)$, $(\exists)M > 0$ such that

$(\forall)$ t>o, for all $x \in X, N_1(x,t) > \beta \Rightarrow$N$(f(x), Mt)$>$\alpha$.

But

N$(f(x), Mt)$>$\alpha \Leftrightarrow$ N$(f(x), Mt)$ =1 $\Leftrightarrow$ $|f(x)| < Mt$.

Hence $(\exists)$ $\beta \in(0,1)$, $(\exists)M > 0$ such that

For all t>o, for all $x \in X, N_1(x,t) > \beta \Rightarrow |f(x)| < Mt$.

Hence proved.

CONCLUSION:

As fuzzy continuity and topological continuity are equivalent and since FNLSs of topological space,all results and theorems in topological linear spaces hold for FNLSs. We can obtain fuzzy version for the classical principles of functional analysis(such as the uniform boundedness principle, the open mapping theorem and the closed graph theorem). This remark was made by I. Sadeqi and F.S. Kia[12]for FNLSs of type(X,N,∧).particularly we have conclude that the theorem of fuzzy continuous and uniform continuity theorem , Based on our resuts.

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