Fixed Point Theory and Its Application

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Abstract

Many problems in pure and applied mathematics have as their solutions the fixed point of some mapping F. Therefore a number of procedures in numerical analysis and approximations theory obtain successive approximations to the fixed point of an approximate mapping. Our object in this paper to discuss about fixed point theory and its applications. The fixed point theory is essential to various theoretical and applied fields, such as variational and linear inequalities, the approximation theory, nonlinear analysis, integral and differential equations and inclusions, the dynamic systems theory, mathematics of fractals, mathematical economics (game theory, equilibrium problems, and optimisation problems) and mathematical modelling. This paper presents a few benchmarks regarding the applications of the fixed point theory. This paper also debates if the results of the fixed point theory can be applied to the mathematical modelling of quality.

Keywords: *Theory, Application, Model, System, Fixed.*

I. INTRODUCTION

The flourishing field of fixed point theory started in the early days of topology with seminal contributions by Poincare, Lefschetz-Hopf, and Leray-Schauder at the turn of the 19th and early 20th centuries. The theory vigorously developed into a dense and multifaceted body of principles, results, and methods from topology and analysis to algebra and geometry as well as discrete and computational mathematics. This interdisciplinary theory par excellence provides insight and powerful tools for the solvability aspects of central problems in many areas of current interest in mathematics where topological considerations play a crucial role. Indeed, existence for linear and nonlinear problems is commonly translated into fixed point problems; for example, the existence of solutions to elliptic partial differential equations, the existence of closed periodic orbits in dynamical systems, and more recently the existence of answer sets in logic programming

The classical fixed point theorems of Banach and Brouwer marked the development of the two most prominent and complementary facets of the theory, namely, the metric fixed point theory and the topological fixed point theory. The metric theory encompasses results and methods that involve properties of an essentially isometric nature. It originates with the concept of Picard successive approximations for establishing existence and uniqueness of solutions to nonlinear initial value problems of the 1st order and goes back as far as Cauchy, Liouville, Lipschitz, Peano, Fredholm, and most particularly, Emile Picard. However, the Polish mathematician Stefan Banach is credited with placing the underlying ideas into an abstract framework suitable for broad applications well beyond the scope of elementary differential and integral equations. Metric fixed point theory for important classes of mapping gained respectability and prominence to become a vast field of specialization partly and not only because many results have constructive proofs, but also because it sheds a revealing light on the geometry of normed spaces, not to mention its many applications in industrial fields such as image processing engineering, physics, computer science, economics, and telecommunications.

A particular interest in fixed points for set-valued operators developed towards the mid-20th century with the celebrated extensions of the Brouwer and Lefschetz theorems by Kakutani and Eilenberg-Montgomery, respectively. The Banach contraction principle was later on extended to multivalued contractions by Nadler. The fixed point theory for multivalued maps found numerous applications in control theory, convex and nonsmooth optimization, differential inclusions, and economics. The theory is also used prominently in denotational semantics (e.g., to give meaning to recursive programs). In fact, it is still too early to truly estimate the importance and impact of set-valued fixed point theorems in mathematics in general as the theory is still growing and finding renewed outlets.

II. APPLICATION OF FIXED POINT THEORY IN DIFFERENT FIELDS

Starting with the multivalued version of the Banach-Caccioppoli contraction principle, demonstrated by S. B. Nadler Jr. in 1969, the fixed point theory for multivalued operators in metric spaces has been used in many works published in the specialty literature. The development of this theory led to the development of various applications in numerous fields, such as: the optimisation theory, integral and differential equations and inclusions, the theory of fractals, econometrics, etc. Among fixed point theorems with multivalued applications, one with remarkable applications is also the Avramescu-Markin-Nadler theorem. Applications of the fixed point theorem that were identified and demonstrated were: Kasahara fixed point theorems, fixed point theorems for Darboux functions, etc. Other important results with many applications to the fixed point theory are the fixed point theorems in fuzzy metric spaces, which are included in many works published in the specialty literature

Fixed point theory is important not only in the existence of the theory of differential equations, integral equations, differential inclusions, integral inclusions, functional equations, partial differential equations, random differential equations, the approximation methods but also in economics and management, in computer science and other domains.

The fixed point theory has applications in many problems, such as the existence of solutions, the existence of orbits in dynamical systems, in programming, etc. The fixed points of certain important single-valued mappings also play an important role, as their results can be applied in engineering, physics, computer science, economics, and in telecommunication

III. FIXED POINT THEORY APPLIED TO THE GAME THEORY

Particular Case – Games for the Field of Quality

The problems related to the analysis of the quality of a tangible or intangible product may be approached, in some cases, as problems from the game theory.

Creating a tangible or intangible product depends on two groups of factors: one group that increases the values of the product's quality indicators, and the other group that decreases the values of the product's quality indicators. Therefore, the first player is determined by the factors that increase the values of the quality indicators, and the second player by the other group of factors. The first player "wishes" to create a high quality product, whereas the second player "wishes" to create a poor quality product. The result of the competition between them is the actual quality of the product.

According to dedicated scientific notation, we denote by

$A1 = \{\alpha1, ..., \alpha i, ...,\alpha m\}$ the set of factors that lead to an increase in the values of the quality indicators, and by

$A2 = \{a1, ..., aj, ...,an\}$ the set of factors that lead to a decrease. At one moment of time from the life cycle of a tangible or intangible product, each player has a certain influence on the values of the quality indicators.

Each player chooses an action αi from A1 and aj from A2. The actions refer to the effect of factor i on the values of the quality indicators.

The utility of choosing action αi by the first player can be described mathematically by a real function $f1(\alpha i, aj)$ and its values can be interpreted as a win for the first player, in this situation. Function $f2(\alpha i, aj)$ represents the second player's loss, in this situation.

According to the specialty literature, the fact that the sum of the game is null can be written as (Owen, 1974):

$$f1(\alpha i, aj) + f2(\alpha i, aj) = 0.$$

The question is how the first player can choose the action αi in order to achieve a maximum gain $f1(\alpha i, aj)$, knowing that the other player has the same objective (the term utility was introduced by von Neumann, and it significantly expanded the concept of "game", suggesting that a "result" of a game is not only something financial, but also a diverse multitude of events for which each player shows interest, quantified by their utility). The Nash Equilibrium in a pure strategy is represented by a strategic profile in which the strategy of each player is the best response to the strategy chosen by the other player. Thus, the conflict situations regarding creating tangible or intangible products of a high quality level and their management can be modelled by using the interconnections between these theories (Figure no.1):

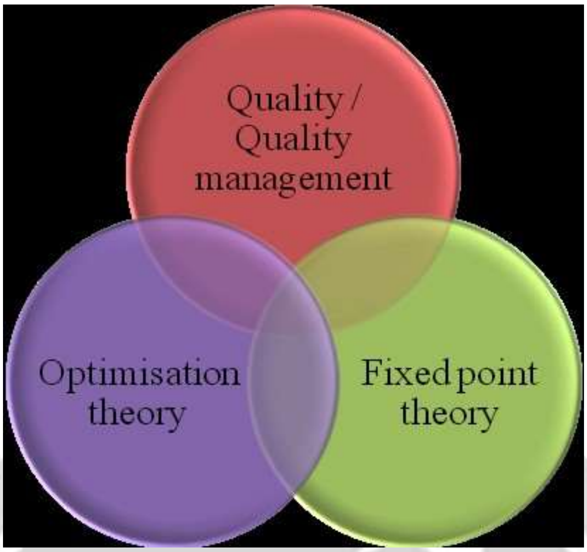

Figure 1: Interconnection between the theories

Several authors have demonstrated that the fixed point theory can be applied to optimisation problems, game theory problems, and also in problems related to the Nash equilibrium.

Example: The quality of the promotion service in universities U1 and U2, in order to increase the number of candidates that wish to enrol in their courses. Based on the data obtained by analysing the number of students that applied in the previous years, the marketing services of the two universities have reached the conclusions shown in the following table:

IV. RESULTS

Table 1: Obtained Results

Strategy of U_1, generated by its set $A_1 = \{\alpha_1, \alpha_2\}$ of action possibilities	Strategy of U_2, generated by its set $A_2 = \{a_1, a_2\}$ of action possibilities	Result (change in the percentage of candidates that wish to enroll to the university)
α_1 (uses flyers)	**a_1 (uses flyers)**	**$f_1(\alpha_i, a_j) = 10$(U_1 wins), $f_2(\alpha_i, a_j) = -10$(U_2 loses)**
α_1 (uses flyers)	**a_1 (uses advertisingmedia)**	**$f_1(\alpha_i, a_j) = 6$(U_1 wins), $f_2(\alpha_i, a_j) = -6$(U_2 loses)**
α_2 (uses advertisingmedia)	**a_2 (uses flyers)**	**$f_1(\alpha_i, a_j) = -12$(U_1 loses), $f_2(\alpha_i, a_j) = 12$(U_2 wins)**
α_2 (uses advertisingmedia)	**a_2 (uses advertisingmedia)**	**$f_1(\alpha_i, a_j) = 2$(U_1 loses), $f_2(\alpha_i, a_j) = -2$(U_2 wins)**

The results from table no. 1 can be summarised as seen in table no. 2:

Table 2: Summary of results

	a_1	a_2	
1	10	6	$min\{10, 6\} - 6$
2	-12	2	$min\{-12, 2\} = -12$
	$max\begin{Bmatrix} 10, \\ 12 \end{Bmatrix} - 10$	$max\{6, 2\} = 6$	$min\{10, 6\} - 6$ $- max\{6, -12\}$

The game has an equilibrium point, (αi, aj), which coincides with the result of the game. University U1 will want to use action αi, meaning to use the flyers, whereas university U2 will try to minimise the loss by choosing action aj.

Games can be cooperative and non- cooperative. In 1986, Kohlberg and Mertens proved that for every finite non- cooperative game, the set of Nash equilibrium points consists of finite components and at least one of them is essential. In 2004, Yu and Yang suggested that essential components can be applied to nonlinear problems. In 2012, Vuong, Strodiot, and Nguyen introduce some new iterative methods for finding a common element of the set of points satisfying a Ky Fan inequality, and the set of fixed points of a contraction mapping in a Hilbert space.

V. CONCLUSIONS

The fixed point theory has had many applications in the last decades. Its applications are very useful and interesting to the optimisation theory, to the game theory, to conflict situations, but also to the mathematical modelling of quality and its management.

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