High Order Iterative methods without derivatives for solving non-leaniar equations

Zabih ullah Zahir,

Student of Mathematics Department, University of College of Science, Osmania University, Hyderabad, India.

Abstract

The new second-order and third-order iterative methods without derivatives are presented for solving nonlinear equa- tions; the iterative formulae based on the homotopy perturbation method are deduced and their convergences are pro- vided. Finally, some numerical experiments show the efficiency of the theoretical results for the above methods.

Keywords: Nonlinear equation; Iterative method; Homotopy perturbation method; Newton method

1. Introduction

In recent years, the scientists and engineers have devoted their attention to the application of the homotopy perturbation method in nonlinear problems, because this method is to continuously deform a difficult problem into a simple problem which is easy to solve. And the homotopy theory becomes a powerful mathematical tool, when it is successfully coupled with perturbation theory [1,7–14,16,17].

Considering the solution x^* of a nonlinear equation

$$f(x)=0 \qquad (1)$$

in [a, b], we have many different methods such as bisection method, fixed-point iteration, secant method, New- ton method, and so on. As is well known, the Newton method is a faster method than the other methods and is quadratically convergent, but computation of $f'(x)$ with $f'(x)$.the higher-orther method are presented interpolation are proposed in [2] but there computational process in complex. in - verse interpolation are proposed in [3] at least three-order method are presented , but they need to compute $\sqrt{f'(x)-2f(x)f''(x)}$ and $(\frac{f(x)}{f'(x)})^2$. In [4-6] the exponential methods with transferring derivative are obtained by using the Liapunov method and Lambert technique , which are quadratically convergent. The iterative method of exponential descent with super-linear convergence and parametric iterative methods of quadratic convergence without the derivative are presented, respectively, but the defect is how to choose the proper parametric values. Moreover, the homotopy techniques were applied to find all roots of a nonlinear equations in [7–15]. And different iterative formulae are derived, but they are needed to compute $f'(x), f''(x)$ *and* $f'''(x)$ .And the convergence rates of these formulae are not given accurately. In this paper, we propose the new second-order and third-order iterative methods based on the homotopy perturbation theory. These methods do not need to compute the derivatives. Thereinto, the second-order iter- ative method has the same asymptotic error constant and the convergence rate compared with the Newton method. And the third-order iterative method has a faster rate of convergence and high precision compared with the Newton method and the new second-order iterative method.

2. The construction of iterative methods

If we wish to find the solution in Eq. (1), we choose an auxiliary function $g(x)=0$. It must be known or controllable and easy to solve. By the homotopy technique, we construct a homotopy $R\times[0.1]\rightarrow R$ which satisfies

$$H(x,p)=pf(x)+(1-p)g(x)=0, \qquad x\in R, \qquad P\in[0,1] \tag{2}$$

where p is an imbedding parameter. Hence, it is obvious that

$$H(x,0)=g(x)=0, \qquad H(x,1)=f(x)=0 \tag{3}$$

and the changing process of p from 0 to 1 is just that of $H(x,p)$ from $f(x)$. In topology, this is called deforma- tion and $g(x)$ and $f(x)$ are considered to be homotopic. Applying the perturbation technique [1,17], due to the fact that $0\le p\le 1$ can be considered as a small parameter, we can assume that the solution of Eq. (2) can be expressed as a series in p

$$x=x_0+px_1+p^2x_2+p^3x_3+.... \tag{4}$$

when $p\rightarrow 1$,Eq.(2) corresponds toEq.(1), (4) becomes the approximate solution of Eq. (1), i.e,

$$x^*=\lim_{p\rightarrow 1}x=x_0+x_1+x_2+x_3+..... \tag{5}$$

To obtain its approximate solution of Eq. (2) we first expand it into Tyler series

$$g(x)=g(x_0)+g'(x_0)\left(px_1+p^2(x_2)+...\right)+\frac{1}{2}g''(x_0)\left(px_1+p^2(x_2)+...\right)^2 \tag{6}$$

$$g(x)=g(x_0)+g'(x_0)\left(px_1+p^2(x_2)+...\right)+\frac{1}{2}g''(x_0)\left(px_1+p^2(x_2)+...\right)^2 \tag{7}$$

Substituting (6) and (7) into (2), and equating the coefficients of like powers of p, we obtain

$$
\begin{aligned}
&p^0 : g\left(x_0\right)=0\\
&p^1 : g'\left(x_0\right)x_1+f\left(x_0\right)-g\left(x_0\right)=0 \qquad (8)\\
&p^2 : g'\left(x_0\right)x_2+[f'\left(x_0\right)-g'\left(x_0\right)]x_1+\frac{1}{2}g''\left(x_0\right)x_1^2=0\\
&p^3 : g'\left(x_0\right)x_3+\frac{1}{2}[f''\left(x_0\right)-g''\left(x_0\right)]x_1+g''\left(x_0\right)x_1x_2+[f'\left(x_0\right)-g'\left(x_0\right)]x_2+\frac{1}{3!}g''\left(x_0\right)x_1^2=0
\end{aligned}
$$

From the above Eqs. (8), if $g'(x_0)\neq 0$, then $x_0 \sim x_3$ can be solved, spectively, x_0 is the exact solution of the auxiliary function $g\left(x\right)=0$ (9)

$$x_1=\frac{-f\left(x_0\right)}{g'\left(x_0\right)}, \tag{10}$$

$$x_2=\frac{\left[g'\left(x\right)-f'\left(x\right)\right]x_1-\frac{1}{2}g''\left(x_0\right)x_1^2}{g'\left(x_0\right)}, \tag{11}$$

$$x_3=\frac{\frac{1}{2}\left[g''\left(x_0\right)-f''\left(x_0\right)x_1\right]+g''\left(x_0\right)x_1x_2-\left[f''\left(x_0\right)-g'\left(x_0\right)x_1\right]x_2-\frac{1}{3!}g'''\left(x_0\right)x_1^3}{g'\left(x_0\right)}, \tag{12}$$

Due to the reason that $g\left(x\right)$, $g'\left(x\right)$, $g''\left(x\right)$ and $g'''\left(x\right)$ must be known or controllable and are easy to be solved, so we can choose $g\left(x\right)$ as the following forms:

1. $g\left(x\right)$ is a linear function, quadratic function, and so on;
2. $g\left(x\right)$ is the part function of $f\left(x\right)$.

Particularly, if we choose the function $g\left(x\right)=f\left(x\right)-f\left(x_0\right)=0$, x_0 is an initial approximation of Eq. (1), then from the above Eqs. (9)–(12), $x_0 \sim x_3$ can be solved, respectively,

$$x_1 = -\frac{f\left(x_0\right)}{g'\left(x_0\right)}, \tag{13}$$

$$x_2 = -\frac{1}{2!}\frac{f''\left(x_0\right)}{f'\left(x_0\right)}x_1^2 = -\frac{1}{2!}\frac{f''\left(x_0\right)}{f'\left(x_0\right)}\left\{\frac{f\left(x_0\right)}{f'\left(x_0\right)}\right\}^2, \tag{14}$$

$$x_3 = -\frac{f''\left(x_0\right)}{f'\left(x_0\right)}x_1 x_2 - \frac{1}{3!}\frac{f'''\left(x_0\right)}{f'\left(x_0\right)}x_1^3, \tag{15}$$

Therefor we can obtain

$x = x_0 + x_1$, with first- order approximation

And

$x = x_0 + x_1 + x_2$, with second- order approximation

And

$x = x_0 + x_1 + x_2 + x_3$, with third - order approximation

We can write done the iteration forms of (13) – (15) as follows:

$$x_{n+1} = x_n - \frac{f\left(x_n\right)}{g'\left(x_n\right)}, \tag{16}$$

$$x_{n+1} = x_n - \frac{f''\left(x_n\right)}{f'\left(x_n\right)} - \frac{f''\left(x_n\right)}{f'\left(x_n\right)}\left\{\frac{f\left(x_n\right)}{f'\left(x_n\right)}\right\}^2 \tag{17}$$

$$x_{n+1} = x_n - \frac{f''\left(x_n\right)}{f'\left(x_n\right)} - \frac{f''\left(x_n\right)}{2f'\left(x_n\right)}\left\{\frac{f\left(x_n\right)}{f'\left(x_n\right)}\right\}^2 + \left\{\frac{f'''\left(x_n\right)}{6f'\left(x_n\right)} - \left[\frac{1}{2}\left[\frac{f''\left(x_n\right)}{f'\left(x_n\right)}\right]^2\right]\right\}\left\{\frac{f\left(x_n\right)}{f'\left(x_n\right)}\right\}^2 \tag{18}$$

Obviously, the iteration formula (16) is the well-known Newton iteration formula. In order to avoid computing derivative, we replace $f'\left(x_n\right)$ and $f''\left(x_n\right)$ by the center difference

$$\frac{f(x_n+h)-f(x_n-h)}{2h}$$

And

$$\frac{f(x_n+h)+f(x_n-h)-2f(x_n)}{h^2}$$

respectively, here $h = af'\left(x_n\right)$ is a step size and $\alpha \neq 0$ is a parameter. So we get the following second-order iter- ative method:

$$x_{n+1} = x_n - \frac{2hf(x_n)}{f(x_n+h)-f(x_n-h)} \tag{19}$$

And the third - order iterative method:

$$\lim_{n\to\infty}\frac{x_{n+1}-x^*}{\left(x_n-x^*\right)^3} = \frac{f''\left(x*\right)^2 - \left(f'\left(x^*\right) - \alpha^2 f'\left(x^*\right)^3\right)f'''\left(x^*\right)/3}{2f'\left(x^*\right)^2} \tag{20}$$

respectively. So, the above iterative formulae do not need to compute $f'(x_n)$ and $f''(x_n)$ in every iteration. It is noticed that the iterative formula (18) is at least cubically convergent if we replace $f'''(x_n)$ by the proper difference. For simplicity, we are not given the strict proof of the convergence rate of iterative formula (18) .

3. The convergent proof of iterative formulae

Theorem 1. Let $f''(x_n)$ be continuous in a sufficiently small neighborhood of x^*; and let $f(x^*)=0$, $f'(x^*)\neq 0$. Then the sequence $\{x_n\}$ produced by the iterative formula (19) is at least quadratically convergent, and the asymptotic error constant is

$$\lim_{n\to\infty}\frac{x_{n+1}-x^*}{(x-x^*)^2}=\frac{f''(x^*)}{2f(x^*)},$$

Proof: Let $e_n = x_n - x^*$, by using Taylors formula, we have

$$f(x_n) = f'e_n + f''e_n^2/2+o(e_n^2),$$

$$f(x_n+h)=(1+\alpha f')f'e_n+(1+3af'+\alpha^2 f'^2)f''e_n^2/2+o(e_n^2),$$

Here $f' = f'(x^*), f'' = f''(x^*)$ therefore, there holds

$$f(x_n+h)-f(x_n-h)=2\alpha f'^2 e_n+3\alpha ff''e_n^2+o(e_n^3)$$

So we have

$$e_{n+1}=e_n-\frac{2hf(x_n)}{f(f(x_n+h)-f(x_n-h))}$$

$$=e_n^2\frac{f''+o(e_n)}{2f'+o(e_n)}$$

namly,

$$\lim_{n\to\infty}\frac{e_{n+1}}{e_n^2}=\frac{f''}{2f'}$$

From Theorem 1, we concluded that the iterative formula (19) has the same asymptotic error constant and the same rate of convergence compared with the iterative formula (16).

Theorem 2. Suppose that $f'''(x)$ be continuous in a sufficiently small neighborhood of x^*, and if f(x) satisfy $f(x^*)=0$, $f'(x^*)\neq 0$, then the sequence $\{x_n\}$ produced by the iterative formula (20) is at least cubically convergent and is of the following asymptotic error constant:

$$\lim_{n\to\infty}\frac{x_{n+1}-x^*}{\left(x_n-x^*\right)^3}=\frac{f''\left(x*\right)^2-\left(f'\left(x^*\right)-\alpha^2 f'\left(x^*\right)^3\right)f'''\left(x^*\right)/3}{2f'\left(x^*\right)^2}$$

Proof: Let $e_n = x_n - x^*$, by using Taylors expansion, we have

$$f\left(x_n\right)= f'e_n + f''e_n^2/2+ f'''e_n^3/6+o\left(e_n^3\right),$$

$$f\left(x_n+h\right)=\left(1+\alpha f'\right)f'e_n+\left(1+3af'+\alpha^2 f'^2\right)f''e_n^3/2$$
$$+\left[\alpha f' f''+3\alpha\left(1+\alpha f'\right)f''^2+\left(1+\alpha f'\right)^3 f'''\right]e_n^3/3+o\left(e_n^3\right)$$

Here $f'=f'\left(x^*\right), f''=f''\left(x^*\right), f'''=f'''\left(x^*\right)$. Therefore, there holds

$$f(x_n+h)-f(x_n-h)=2\alpha f'^2 e_n+3\alpha ff''e_n^2+\left(3\alpha f''^2+4\alpha ff''+\alpha^3 f'^3 f'''\right)e_n^3/3+o\left(e_n^3\right)$$

$$f(x_n+h)+f(x_n-h)-2\alpha f'^2 e_n=\alpha^2 f'^2 f''e_n^2+\alpha^2\left(ff''^2+f'^2 f'''\right)e_n^3+o\left(e_n^3\right)$$

By using a similar method to that used in the proof of Theorem 1, we have

$$e_{n+1}=e_n-\frac{2hf\left(x_n\right)}{f\left(f(x_n+h)-f(x_n-h)\right)}-\frac{4hf\left(x_n\right)^2\left[f\left(f(x_n+h)-f(x_n-h)\right)-2f\left(x_n\right)\right]}{\left[f\left(f(x_n+h)-f(x_n-h)\right)\right]^3}$$
$$=e_n^2\frac{f''^2-\left(f'-\alpha^2 f'^3\right)f'''/3+o\left(e_n\right)}{2f'+o\left(e_n\right)}$$

namly,

$$\lim_{n\to\infty}\frac{e_{n+1}}{e_n^2}=\frac{f''^2-\left(f'-\alpha^2 f'^3\right)f'''/3}{2f'^2},$$

compared with the iterative formulae (16) and (19) .

4. Numerical experiments

By finding the solution x^* of the nonlinear equation $f\left(x\right)=0$ in a given interval $\left[a,b\right]$, we compare the iter- ative methods (19) and (20) with the Newton method (16).

Starting with the same initial value x_0, we compute the approximation x_{n+1} of solution x^* in a given interval $\left[a,b\right]$.The stopping criterion is $\frac{\left|x_{n+1}-x_n\right|}{\left|x_n\right|}<10^{-6}$ and $\left|f\left(x_{n+1}\right)\right|<10^{-6}$.For the convenience of comparison, we choose the same parameter values in the iterative formulae (19) and (20) (see Tables 1–8).

Table 1
The iterations of solving Example 1

Table 1
The iterations of solving example 1

Initial values x_0	Formula (16)	Parameter value α	Formula (19)	Formula (20)
$\pi/3$	5	0.1	5	4
$-\pi/6$	5	0.1	5	4
$\pi/5$	4	0.1	4	3

Table 2
The iterations of solving example 2

Initial values x_0	Formula (16)	Parameter value α	Formula (19)	Formula (20)
5	Divergent	0.01	Divergent	6
1	Divergent	0.01	Divergent	7
4	4	0.01	4	4

Table 3
The iterations of solving example 3

Initial values x_0	Formula (16)	Parameter value α	Formula (19)	Formula (20)
5	27	0.1	10	5
0.1	4	0.1	4	3
1	4	0.1	4	3

Table 4
The iterations of solving example 4

Initial values x_0	Formula (16)	Parameter value α	Formula (19)	Formula (20)
4	5	0.1	5	3
20	5	0.1	5	3
10	4	0.1	4	2

Table 5
The iterations of solving example 5

Initial values x_0	Formula (16)	Parameter value α	Formula (19)	Formula (20)
4	Divergent	0.1	Divergent	5
0.5	5	0.1	5	3
1.5	5	0.1	5	3

Table 6
The iterations of solving example 6

Initial values x_0	Formula (16)	Parameter valueα	Formula (19)	Formula (20)
2	11	0.001	11	8
1	8	0.001	8	7
1.2	5	0.001	5	4

Table 7
The iterations of solving example 7

Initial values x_0	Formula (16)	Parameter valueα	Formula (19)	Formula (20)
2	7	0.01	7	5
-1	9	0.01	9	6
1	5	0.01	5	4

Table 8
The iterations of solving example 8

Initial values x_0	Formula (16)	Parameter valueα	Formula (19)	Formula (20)
1.5	5	0.1	10	5
1	4	0.1	4	3
0.4	6	0.1	4	3

Example 1

$$f(x) = 2\sin x - 1 \quad , x \in \left[-\frac{\pi}{6}, \frac{\pi}{3}\right], x^* = 0.52359877559830$$

Example 2

$$f(x) = e^{\sin x} - x - 1 \quad , x \in [1,4], x^* = 1.69681238680975$$

Example 3

$$f(x) = \arctan x + \sin x + x - 2 \quad , x \in [0.1,5], x^* = 0.71858676906358$$

Example 4

$$f(x) = \sqrt{x-2} - 3, x \in [4,20], x^* = 11.$$

Example 5

$$f(x) = \ln x, x \in [0.5,4], x^* = 1.$$

Example 6

$$f(x) = x^{10} - 2x^3 - x + 1, x \in [1,2], x^* = 1.11033918535812$$

Example 7

$$f(x) = x^2 - (1-x)^5 \quad , x \in [-1,2], x^* = 0.34595481584824$$

Example 8

$$f(x) = e^x - 3x^2 \ , x \in [0.4, 1.5], x^* = 0.91000757248871.$$

The above numerical results show that the iterative formulae (19) and (20) without computing derivatives are at least quadratically convergent and maintain the convergent and computational efficiency. In many cases, compared with the Newton method, the iterative times are relatively less and the rate of convergence is faster.

Conclusion

In this work we present a new higher-order iterative method for solving non-linear equation which requires two function and higher- order derivatives evaluations per step, it was compared in its performance to some higher-order methods, and the proposed method has been observed to have at least better performance and more stability.

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