INTEGRO DIFFERENTIAL FRACTIONAL BOUNDARY VALUE PROBLEM ON AN UNBOUNDED DOMAIN ALONG WITH EQUATIONS IN BANACH SPACES

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ABSTRACT

In this paper we discuss about the existence of solutions for non linear fractional differential equations of volterra type and non local integro differential conditions on an infinite interval by using fixed point method. We describe some main results which can be used to prove theorems.

KEYWORDS

Fractional integro-differential equations, Boundary value problem, equations of volterra type, unbounded domain, fixed point, Riemann-Liouville fractional derivatives.

1.Introduction

Fractional calculus was originated and it has gained much attention in recent years by many researchers.

Fractional differential equations appears in a large number of fields of science and engineering, thermodynamics,elasticity, electric railway systems, telecommunication lines and also in chemistry, analysing kineticalreaction problems.

In addition, scientists have found that many mathematics models can be reduced to the nonlocal problems with integral boundary conditions.

In the past decades, nonlocal boundary value problems of fractional differential equations on finite or infinite interval have been extensively investicated; see, for instance, however, to the best of our knowledge, very little is known regarding integro differential fractional boundary value problem on an infinite interval.

In this paper, we consider the following integro differential fractional boundary value problem for nonlinear fractional differential equations of Volterra type on an infinite interval.

$$D^{\alpha}u(t) + f\big(t, u(t), Tu(t)\big) = 0,\ \ 3 < \alpha \le 4,$$

$$u(0) = u'(0) = u''(0) = 0, D^{\alpha-1}u(\infty) = \xi I^{\beta}u(\sigma), \quad \beta > 0,$$

Where $t \in J = [0, +\infty), f \in C\ [J \times R \times R, R], \xi \in R, \sigma \in J,$

D^{α} is the Riemann Liouville fractional derivative of order α ,I^{β} is theRiemann Liouville fractional integral of orderβ,and $(Tu)(t) = \int_0^t K(t,s)\,u(s)ds$ with $(t,s) \in\ C\ [D,R], D = \{(t,s) \in R^2 | 0 \le s \le t\}$.

Define the space $X = \left\{u \in C(J,R) : \sup_{t\in J} \frac{|u(t)|}{1+t^{\alpha_1}} < +\infty\right\}$equipped with the norm

$\|u\|_X = \sup_{t\in J} \frac{|u(t)|}{1+t^{\alpha_1}}$It is obvious that$X$ is a Banach space.

2.Preliminaries

In this section, we first present some useful definitions.

Definition 2.1

The Volterra Integral equations are a special type of integral equations.They are divided into two groups referred to as the first and the second kind.

A linear Volterra equation of the first kind is

$f(t) = \int_a^t K(t,s)x(s)ds$.Where f is a given function and x is an unknown function.

A linear Volterra equation of the second kind is

$$x(t) = f(t) + \int_a^t K(t,s)x(s)ds.$$

Definition 2.2

Let x, y be a Banach spaces. An unbounded operator (or simplyoperator) $T: x \to y$ is a linear map T from a linear subspace $D(T) \subseteq x$.

The domain T to the space y. T may not be defined on the whole space $x. Two$ operators are equal if they have a common domain and they coincide on that common domain.

An operator T is said to be closed if its graph $\Gamma(T)$ is a closed set.

[Here the graph $\Gamma(T)$ is a linear subspace of the direct sum $x \oplus y$, definedas the set of all pairs (x, Tx), where x runs over the domain of T].

This means that for every sequence $\{x_n\}$ of points from the domain of T such that$x_n \longrightarrow x$and $Tx_n \longrightarrow y$. It holds that x belongs to the domain of $T and\ Tx = y$.

The closedness can also be formulated interms of the graph norm an operator T is closed if and only if its domain$D(T)$ is a complete space with respect to the norm.

$$\|x\|_T = \sqrt{\|x\|^2 + \|Tx\|^2}.$$

Definition 2.3

A fixed point of a function, f is any value x for which $(x) = x$. A function may have any number of fixed points from none $(eg\ f(x) = x + 1)$ to infinitely many $f(x) = x$.

The fixed point combinatory, written as either "fix" or "Y" will return the fixed point of a function.

Definition2.4

The Riemann Liouville fractional derivative of order α for a continuous function f is defined by

$D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_0^t(t-s)^{n-\alpha-1}f(s)ds, n=[\alpha]+1$,provided that the right hand side is pointwise defined on $(0,\infty)$.

Definition 2.5

The Riemann Liouville fractional integral of order α for a function f is defined by

$I^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds, \alpha>0$,provided that such an integral exists.

Definition 2.6

Inmathematics,in the field of differential equations, a boundary value problem is a differential equation together with a set of additional constrains, called the boundary conditions.

A Boundary value problem is a system of ordinary differential equation or a partial differential equation with solution and derivative values specified at more than one point.

Most commonly, the solution and derivatives are specified at just two points (the boundaries) defining a two point boundary value problem.

For example

$$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}-\nabla^2 u=f & in\Omega\\ u(0,t)=u_1 & on\partial\Omega\\ \dfrac{\partial u}{\partial t}(0,t)=u_2 & on\partial\Omega,\end{cases}$$

where$\partial\Omega$ denotes the boundary value of Ω, is a boundary problem.A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions.

3.Result

Let $U\subset$ X be an open bounded set. Then U is relatively compact in X if the following conditions hold.

(i)for any $u(t)\epsilon U$, $\frac{u(t)}{1+t^{\alpha-1}}$is equicontinuous on any compact interval of J

(ii)for any $\varepsilon<0$, there exists a constant $T=T(\varepsilon)>0$ such that

$\left|\frac{u(t_1)}{1+t_1^{\alpha-1}}-\frac{u(t_2)}{1+t_2^{\alpha-1}}\right|<\varepsilon$for any $t_1,t_2\geq T$ and $u\in U$

(K_1) $\xi\geq 0,\Gamma(\alpha+\beta)>\xi\sigma^{\alpha+\beta-1}$.

(K_2) There exists a constantk^* such that $k^*=sup_{t\in J}\int_0^t|k(t,s)|\,(1+s^{\alpha-1})ds<\infty$.$(K_3)$ There exist a nonnegative functions $a(t),b(t),c(t)$ defined on $[0,\infty)$ and constants $p,q\geq 0$ such that

$|f(t,u,v)|\leq a(t)+b(t)|u|^p+c(t)|v|^qand\int_0^{+\infty}a(t)dt=a^*<+\infty$, (K_3) $\int_0^{+\infty}b(t)(1+t^{\alpha-1})^p dt=b^*<+\infty$, $\int_0^{+\infty}c(t)dt=c^*<+\infty$.

Lemma 3.1

If conditions (K_2) and (K_3) are satisfied,then we have$\int_0^{+\infty}|f,u(s),Tu(s)|\,ds\leq a^*+b^*\|u\|_X^p+\ c^*(k^*)^q\|u\|_X^q$, $\forall u\in X$.

Proof:

For all $u \in X$, by conditions ,$(K_2)$and(K_3) , we have

$$\int_0^{+\infty} |f(s,u(s),Tu(s)|\,ds \le \int_0^{+\infty} [a(s) + b(s)|u(s)|^p + c(s)|Tu(s)|^q]\,ds$$

$$\le a^* + \int_0^{+\infty} b(s)\,(1+s^{\alpha-1})^p \frac{|u(s)|^p}{(1+s^{\alpha-1})^p}\,ds \frac{|u(s)|^p}{(1+s^{\alpha-1})^p} + \int_0^{+\infty} c(s)\left[\int_0^s |K(s,r)u(r)|dr\right]^q ds$$

$$\le a^* + b^*\|u\|_X^p + \int_0^{+\infty} c(s)\left[\int_0^s |K(s,r)|(1+r^{\alpha-1})\frac{|u(r)|}{(1+r^{\alpha-1})}\,dr\right]^q ds$$

$\le a^* + b^*\|u\|_X^p + \int_0^{+\infty} c(s)(k^*)^q\,\|u\|_X^q ds.$

$\le a^* + b^*\|u\|_X^p + c^*(k^*)^q\|u\|_X^q$.

The proof is complete.

Theorem 3.2

Let $h \in C([0,+\infty)) with \int_0^\infty h(s)ds < \infty. If \Gamma(\alpha+\beta) \neq \xi\sigma^{\alpha+\beta-1}$, then the fractional integral boundary value problem $D^\alpha u(t) + h(t) = 0,\ u(0) = u'(0) = u''(0) = 0, D^{\alpha-1}u(\infty) = \xi I^\beta u(\sigma),\ \ \beta > 0$,has a unique solution.

Proof:

$u(t) = \int_0^{+\infty} G(t,s)h(s)ds$,WhereG(t,s) = $\frac{1}{\Delta}$ { $[\Gamma(\alpha+\beta) - \xi(\sigma-s)^{\alpha+\beta-1}]t^{\alpha-1}\ -\ [\Gamma(\alpha+\beta) - \xi\sigma^{\alpha+\beta-1}](t-s)^{\alpha-1}$, $s \le t, s \le \sigma, \left[\Gamma(\alpha+\beta) - \xi(\sigma-s)^{\alpha+\beta-1}\right]t^{\alpha-1}, 0 \le t \le s \le \sigma, \Gamma(\alpha+\beta)[t^{\alpha-1} - (t-s)^{\alpha-1}] + \xi\sigma^{\alpha+\beta-1}(t-s)^{\alpha-1}, 0 \le \sigma \le s \le t, \Gamma(\alpha+\beta)t^{\alpha-1},\quad s \ge t, s \ge \sigma.$
and$\Delta = \Gamma(\alpha)[\Gamma(\alpha+\beta) - \xi\sigma^{\alpha+\beta-1}]$.$u(t) = \int_0^{+\infty} G(t,s)\,h(s)ds$ $=$ $-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \frac{\Gamma(\alpha+\beta)t^{\alpha-1}}{\Gamma(\alpha)[\Gamma(\alpha+\beta)-\xi\sigma^{\alpha+\beta-1}]}\left[\int_0^\infty h(s)ds - \int_0^\sigma \frac{\xi(\sigma-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\,h(s)ds\right]$.Then, it is easy to get that $u(0) = u'(0) = u''(0) = 0,$

Multiplying in both sides $D^{\alpha-1}$we get,
$D^{\alpha-1}u(t) =$
$D^{\alpha-1}\left(\int_0^{+\infty} G(t,s)h(s)ds\right)$ $= D^{\alpha-1}(-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \frac{\Gamma(\alpha+\beta)t^{\alpha-1}}{\Gamma(\alpha)[\Gamma(\alpha+\beta)-\xi\sigma^{\alpha+\beta-1}]}\left[\int_0^\infty h(s)ds - \int_0^\sigma \frac{\xi(\sigma-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\,h(s)ds\right])$ $= -\int_0^t h(s)\,ds + \frac{\Gamma(\alpha+\beta)}{[\Gamma(\alpha+\beta)-\xi\sigma^{\alpha+\beta-1}]}\left[\int_0^\infty h(s)ds - \int_0^\sigma \frac{\xi(\sigma-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\,h(s)ds\right]andI^\beta u(t) = I^\beta\left(\int_0^{+\infty} G(t,s)h(s)ds\right)$
$= I^\beta(-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \frac{\Gamma(\alpha+\beta)t^{\alpha-1}}{\Gamma(\alpha)[\Gamma(\alpha+\beta)-\xi\sigma^{\alpha+\beta-1}]}\left[\int_0^\infty h(s)ds - \int_0^\sigma \frac{\xi(\sigma-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\,h(s)ds\right])$ $=$
$-\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} h(s)\,ds + \frac{t^{\alpha+\beta-1}}{[\Gamma(\alpha+\beta)-\xi\sigma^{\alpha+\beta-1}]}\left[\int_0^\infty h(s)ds - \int_0^\sigma \frac{\xi(\sigma-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\,h(s)ds\right]$.Thus, we can get the relation , $D^{\alpha-1}u(\infty) = \xi I^\beta u(\sigma)$.Given equation,
$D^\alpha u(t) + h(t) = 0$,That is, $D^\alpha u(t) = -h(t)$Finally, applying$u(t)$ value,by a simple deduction it follows$D^\alpha u(t) = D^\alpha(-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \frac{\Gamma(\alpha+\beta)t^{\alpha-1}}{\Gamma(\alpha)[\Gamma(\alpha+\beta)-\xi\sigma^{\alpha+\beta-1}]}\left[\int_0^\infty h(s)ds - \int_0^\sigma \frac{\xi(\sigma-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\,h(s)ds\right]$) $= -h(t)$.

The proof is complete.

Theorem 3.3

Let X be a Banach space, T and S commutative, continuous mappings from X into X such that for every $y \in X$ there exists one and only $x(y) \in X$such that $x(y) \in X$ such that $x(y) = Tx(y) + y$ and the following conditions are satisfied: (1) The mapping S is additive,locally compact and $\overline{s^n}(x)$ is compact for some n$>$ 1. (2)The mapping $R: y \to x(y)$ is continuous on X.Then Fix (T+S)$\neq \emptyset$.**Proof:**

First we shall prove that for every $x \in X$. (1)$RSx = SRx$Using the relation $Rx = TRx + x$for every $x \in X$.

It follows that$SRx = S(TRx + x) = STRx + Sx = T(SRx) + Sx$and since the equation$z = Tz + Sx$ has one and only one solution RS. It follows that the relation(1) holds.

Let us define the mapping $R^*: X \to X$ in the following way$R^*x = RSx$ for every $x \in X$.

Let X be a banach space and $F: X \to X$ be a continuous mapping so that $F^n(X)$is relatively compact for some$n > 1$. If the mapping F is locally compact thenFix(F) # ∅ are satisfied.

Since, R is continuous and S is locally compact.It follows that, R^* is locally compact.

Further, $(R^*)^n = R^n S^n$ and so the set$(R^*)^n(X)$ is relatively compact.

Let X be a banach space and $F: X \to X$ be a continuous mapping so that $F^n(X)$is relatively compact for some $n > 1$. If the mapping F is locally compact then Fix(F) # ∅.

We conclude that

Fix(R^*)#$\emptyset$.Further it is obvious that

Fix$(R^*) \subseteq Fix(T + S)$.

Conclusion

We have discussed the existence of solutions for non linear fractional differential equations of volterra type and non local integro differential conditions on an infinite interval by using fixed point method. We describesomemainresultswhichcanbe used toprovetheorems.

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