INTERVAL CRITERIA FOR
OSCILLATION OF SECOND
ORDER NON-LINEAR NEUTRAL
DELAY DIFFERENTIAL
EQUATIONS

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Abstract

Oscillation criteria are established in this paper for the second order non-linear neutral delay
differential equations

\[ \left[ w(t) \left( x(t) + m(t) x(t - \tau) \right) \right]' + n(t) f(x(t - \sigma)) = 0 \]

Where \( \tau \) and \( \sigma \) are nonnegative constants, \( w, m, n \in C([t_0, \infty), \mathbb{R}) \), and \( f \in C(\mathbb{R}, \mathbb{R}) \). These results are different from known one in the sense that they are based on the information only on a sequence of subintervals of \( [t_0, \infty) \)

Keywords – Neutral differential equations, Oscillation, Criteria, Delay differential equations.

INTRODUCTION

Consider the second order neutral delay differential equation

\[ \left[ w(t) \left( x(t) + m(t) x(t - \tau) \right) \right]' + n(t) f(x(t - \sigma)) = 0 \]  ...  [A]

Where \( t \geq t_0, \tau \) and \( \sigma \) are nonnegative constants,

\[ m, n \in C([t_0, \infty), \mathbb{R}), \text{ and } f \in C(\mathbb{R}, \mathbb{R}) \]. Let us assume the following as

\[ q(t) \geq 0, w(t) > 0, \left( \frac{1}{w(s)} \right) ds = \infty, \]

\[ f(x)/x \geq \gamma > 0 \quad \text{for} \ x \neq 0. \]

THEOREM- 1

If (a) \( n(t) \geq 0, w(t) > 0, \left( \frac{1}{w(s)} \right) ds = \infty, f(x)/x \geq \gamma > 0 \) \( \text{for} \ x \neq 0. \) .... [a]
Holds and \( x(t) \) eventually positive solution of equation (A), then \( z(t) \geq 0, \ z'(t) \geq 0, \)
\[
(w(t)z'(t))' \leq \sigma \text{ on interval } [T_0, \infty)
\]
For some \( T_0 \geq t_0 \) sufficiently large.

Moreover,

(i) If \( 0 \leq m(t) \leq 1, \) then
\[
(w(t)z'(t))' + \gamma n(t)(1 - m(t - \delta))z(t - \delta) \leq 0 \ 
\]
(ii) If \( -1 < m(t) \leq 0 \) then
\[
(w(t)z'(t))' + \gamma n(t)z(t - \delta) \leq 0 \ 
\]

**PROOF:**

Without loss of generality assume that \( x(t) > 0 \) for all \( t \geq T_0 - \tau - \delta. \) Since \( x(t) \geq 0 \) equation (A) implies that \( (w(t)z'(t))' \leq 0 \) and \( (w(t)z'(t)) \) is decreasing.

It follows that
\[
limit_{t \to \infty} w(t)z'(t) = 1
\]
Let us prove that
\[
w(t)z'(t) \geq 0
\]
And[1.1] and[1.2] holds.

(i)
If \( 0 \leq m(t) \leq 1, \) then prove that \( w(t)z'(t) \geq 0 \)

Otherwise there exist \( t_1 \geq T_0 \) such that \( z'(t_1) < 0. \)

**From** \( (w(t)z'(t))' \leq 0. \) It follows that
\[
z(t) \leq z(t_1) + w(t_1)z'(t_1) \int_{t_1}^{t} \frac{1}{r(s)} \ ds.
\]
Hence by condition (a) we have \( \lim_{t \to \infty} z(t) = -\infty \) which contradict that \( z(t) > 0 \) for \( t \geq t_0. \)

Now observe that from (A) we have
\[
[w(t)z'(t)'] + n(t)f(x(t - \delta)) = 0. \ 
\]
From the condition
\[
n(t) \geq \sigma, w(t) > 0, \ (1/w(s)) \ ds = \infty, \ f(x)/x \geq \gamma > 0 \text{ for } x \neq 0.
\]
And by[1.3] we get
\[
[w(t)z'(t)'] + \gamma n(t)[x(t - \delta) - m(t - \delta)x(t - \tau - \delta)] \leq 0
\]
Which in view of the fact that \( z(t) \geq x(t) \) and

If \( z(t) \) is increasing we get
\[
(w(t)z'(t))' + \gamma n(t)(1 - m(t - \delta))z(t - \delta) \leq 0
\]
(ii) If \(-1 < \alpha \leq m(t) \leq 0\) then prove that

\[
\lim_{t \to \infty} x(t) = 1 \geq 0.
\]

Otherwise \(1 < 0\) then we get \(\lim_{t \to \infty} z(t) = -\infty\) which claims that \(z(t)\) cannot be eventually negative on \([T_0, \infty)\)

*If that is the case it can consider two mutually exclusive cases.*

**CASE 1:**

\(x(t)\) is unbounded then there exist an increasing sequence \(\{t_k\}, t_k \to \infty, k \to \infty\)

Such that

\[
x(t) = \sup_{t \leq t_k} x(t) \text{ and } x(t_k) \to \infty \text{ as } t_k \to \infty.
\]

We find that

\[
z(t_k) = x(t_k) + m(t_k) x(t_k - \tau) \geq x(t_k) [1 + m(t_k)] \geq 0
\]

Contradicts the fact that \(\lim_{t \to \infty} z(t) = -\infty\)

**CASE 2:**

\(x(t)\) is bounded then there exist an sequence \(\{t_k\}\) such that

\[
\lim_{k \to \infty} x(t_k) = \lim_{t \to \infty} x(t).
\]

Since the sequence \(\{x(t_k - \tau)\}\) and \(\{m(t_k)\}\) and bounded, there exist convergent subsequences. Therefore, without loss of generality, we may suppose that

\[
\lim_{k \to \infty} (t_k - \tau) \text{ and } \lim_{k \to \infty} m(t_k)
\]

exist.

Hence,

\[
0 > \lim_{k \to \infty} z(t_k) = \lim_{k \to \infty} [x(t_k) + m(t_k) x(t_k - \tau)]
\]

\[
\geq \lim_{k \to \infty} [x(t_k) + m(t_k) x(t_k)]
\]

\[
\geq \lim_{t \to \infty} \sup_{k \to \infty} x(t) [1 + \lim_{k \to \infty} m(t_k)]
\]

\[
\geq 0.
\]

This is also a contradiction.

Thus we must have \(1 \geq 0\), which implies that \(z(t)\) must be eventually positive.

i.e.) there exist \(t_0 \geq t_0\) such that \(z(t) > 0\) for all \(t \geq t_0\). Otherwise since \(\lim_{t \to \infty} w(t) x(t) = 1 \geq 0\), and \(w(t) x(t)\) is nonincreasing, we must have \(z(t) < 0\) for some \(t \geq t_0\).

We therefore have

\[
z(t) < 0, z'(t) \geq 0, (w(t) \alpha'(t))' \leq 0
\]

on \([T_0, \infty)\) for some \(T_0 \geq t_0\) sufficiently large.

From condition (a), we have \(f(x(t) - \delta) \geq \gamma x(t - \delta) \geq \gamma z(t - \delta)\) for \(t \geq t_0\) sufficiently large. And we find equation (A) implies the equation for \(z(t)\).
Hence proved.

RESULT:

In theorem 1 if \( x(t) \) is an eventually negative solution of (A), then the relevant result hold.

In the sequence we say that a function \( H = H(t, s) \) belongs to function class \( K \), denoted \( HEK \), if
\[
H \in C(D, R_{+} = (0, \infty)) \quad \text{and} \quad k \in C^{1}(D, R_{+})
\]

Where \( D = \{(t, s); -\infty < s < t < \infty\} \) which satisfies
\[
H(t, t) = 0, H(t, s) > 0, \quad \text{for} \ t > s, \quad \text{.......... (B1)}
\]

And has partial derivatives
\[
\frac{\partial H(t, s)}{\partial t} \quad \text{and} \quad \frac{\partial H(t, s)}{\partial s} \quad \text{on} \ D \quad \text{such that}
\]
\[
\frac{\partial}{\partial s} \left( H(t, s)k(t) \right) = h_{1}(t, s)\sqrt{(H(t, s)k(t))}.
\]
\[
\frac{\partial}{\partial s} \left( H(t, s)k(t) \right) = -h_{2}(t, s)\sqrt{(H(t, s)k(t))}, \quad \text{.......... (B2)}
\]

Where \( h_{1}, h_{2} \in C(D, R) \).

THEOREM-2

If (a)
\[
n(t) \geq 0, w(t) > 0, \quad x(t) = \left( \frac{1}{w(s)} \right) ds = \infty, f(x) / x \equiv \gamma > 0
\]

For \( x(t) \neq 0 \) holds and \( x(t) \) be a solution of (A) such that \( x(t) \neq 0 \) for some \( T_{0} \). For any \( g \in C^{1}(T_{0}, \infty, R) \) let
\[
r(t) = -\nu(t) \left( \frac{w(t)x'(t)}{x(t) - \delta} + w(t) - \delta \right) \quad \text{.......... [2.1]}
\]

Where \( t \in [T_{0}, \infty) \). Then for any \( H \in K \).

(i). If \( 0 \leq m(t) \leq 1 \) and \( t \in [c, b] \subset [T_{0}, \infty) \) then
\[
\int_{c}^{b} H(b, s) \phi_{1}(s) ds \leq -H(b, c)k(c)w(c) + \frac{1}{4} \int_{c}^{b} w(s - \delta)\nu(s)h_{2}^{2}(t, s) ds.
\]

(ii). If \( -1 < a < m(t) \leq 0 \) and \( t \in [c, b] \subset [T_{0}, \infty) \) then
\[
\int_{c}^{b} H(b, s) \phi_{2}(s) ds \leq -H(b, c)k(c)r(c) + \frac{1}{4} \int_{c}^{b} w(s - \delta)\nu(s)h_{2}^{2}(t, s) ds.
\]

PROOF
CASE (i)

Without loss of generality assume that \( x(t) > 0 \) for all \( t \geq T_0 + \tau - \delta \)

Differentiating [2.1] and make use of (A) and by theorem (1.) Case: (i), we get that for \( s \in [c, b) \)

\[
r'(s) = 2g(s)r(s) - v(s) \left( \frac{(w(s)x'(s))'}{x(s-\delta)} - \frac{w(s)x'(s)x'(s-\delta)}{x^2(s-\delta)} + [w(s-\delta)g(s)]' \right) \\
\geq 2g(s)r(s) + v(s) \left( \gamma n(s)[1 - m(s - \delta)] + \frac{w(s)x'(s)x'(s-\delta)}{x^2(s-\delta)} - [w(s-\delta)g(s)]' \right)
\]

From the fact that \( w(s)x'(s) \) is decreasing, we get
\[w(s)x'(s) \leq w(s-\delta)x'(s-\delta)\]

For \( s \geq T_0 \)

From the above we know that
\[
r'(s) \geq 2g(s)r(s) + v(s) \left( \gamma n(s)[1 - m(s - \delta)] + \frac{1}{w(s-\delta)} \left( \frac{w(s)x'(s)}{x(s-\delta)} \right)^2 - [w(s-\delta)g(s)]' \right) \\
= 2g(s)r(s) + v(s) [w(s-\delta)g(s)]' + v(s) \left( \gamma n(s)[1 - m(s - \delta)] + \frac{1}{w(s-\delta)} \left( \frac{w(s)}{v(s)} - w(s-\delta)g(s) \right)^2 \right) \\
= \varphi_4(s) + \frac{1}{w(s-\delta)v(s)}r^2(s).
\]

It follows that
\[
\varphi_4(s) \leq r'(s) - \frac{1}{w(s-\delta)v(s)}r^2(s). \tag{2.2}
\]

Multiplying [2.2] by \( H(t,s)k(s) \),

And integrating it with respect to \( s \) from \( c \) to \( t \) for \( t \in [c, b) \), and using the result of (B1) and (B2), one can get
\[
\int_c^t \int_c^s H(t,s)k(s)\varphi_4(s)ds \leq \int_c^t \int_c^s H(t,s)k(s)r'(s)ds - \int_c^t \int_c^s \frac{1}{w(s-\delta)v(s)}H(t,s)k(s)r'(s)ds \\
= -H(t,c)k(c)r(c) + \int_c^t h_2(t,s)\sqrt{H(t,s)k(s)r(s)}ds - \int_c^t \frac{H(t,s)k(s)}{w(s-\delta)v(s)}r^2(s)ds \\
= -H(t,c)k(c)r(c) - \int_c^t \left[ \frac{H(t,s)k(s)}{w(s-\delta)v(s)}r(s) - \frac{1}{2w(s-\delta)v(s)}h_2(t,s) \right]^2 ds + \frac{1}{4} \int_c^t w(s-\delta)v(s)h_2^2(t,s)ds \\
\leq -H(t,c)k(c)r(c) + \frac{1}{4} \int_c^t w(s-\delta)v(s)h_2^2(t,s)ds.
\]
Letting $t \to b^-$ in the above (i) is proved.

\[
\int_c^b H(b, s) \varphi_1(s) \, ds \leq -H(b, c) k(c) r(c) + \frac{1}{4} \int_c^b w(s - \delta) v(s) h_2^2(t, s) \, ds.
\]

**CASE (ii)**

Without loss of generality assume that $x(t) > 0$ for all $t \geq T_0 - \tau - \delta$.

Differentiating \([2.1]\) and make use of \(A\) and by theorem \((1)\)

Case: (ii) we get that for $s \in [c, b)$

\[
r'(s) = 2g(s) r(s) - v(s) \left\{ \frac{w(s)}{z(s - \delta)} - \frac{w(s) z'(s)}{z(s - \delta)} (s - \delta) \right\} + [w(s - \delta) g(s)']
\geq 2g(s) r(s) + v(s) \left\{ \frac{1}{w(s - \delta)} \frac{w(s)}{z(s - \delta)} z'(s) - [w(s - \delta) g(s)'] \right\}
\]

Similar to the proof of theorem \((1)\) we can show the following inequality:

\[
\phi_2(s) \leq r'(s) - \frac{1}{w(s - \delta) v(s)} r^2(s).
\]

Multiplying \([2.3]\) by $H(t, s) k(s)$

And integrating it with respect to $s$ from $c$ to $t$ for $t \in [c, b)$,

\[
\int_c^t H(t, s) k(s) \phi_2(s) \, ds \leq \int_c^t H(t, s) k(s) r'(s) \, ds - \int_c^t \frac{1}{w(s - \delta) v(s)} H(t, s) k(s) r^2(s) \, ds
\]

\[
= -H(t, c) k(c) r(c) + \int_c^t H(t, s) k(s) r(s) \, ds - \int_c^t \frac{H(t, s) k(s)}{w(s - \delta) v(s)} r^2(s) \, ds
\]

\[
= -H(t, c) k(c) r(c) - \int_c^t \left[ \frac{H(t, s) k(s)}{w(s - \delta) v(s)} r(s) - \frac{1}{2} \sqrt{w(s - \delta) v(s)} h_2(t, s) \right] \, ds + \frac{1}{4} \int_c^t w(s - \delta) v(s) h_2^2(t, s) \, ds
\]

\[
\leq -H(t, c) k(c) r(c) + \frac{1}{2} \int_c^t w(s - \delta) v(s) h_2^2(t, s) \, ds.
\]

Letting $t \to b^-$ in the above (ii) is proved.

\[
\int_c^b H(b, s) \varphi_2(s) \, ds \leq -H(b, c) k(c) r(c) + \frac{1}{4} \int_c^b w(s - \delta) v(s) h_2^2(t, s) \, ds.
\]

Hence proved.
CONCLUSION:

Throughout this work, we discussed some definition and theorems on Interval criteria for oscillation of second order non-linear neutral delay differential equations and then we discussed for the oscillation of second order non-linear neutral delay differential equations with non-negative constants on the interval $[t_0, \infty).$ Finally we establish that, when the co-efficient of neutral delay differential equations is zero so that the solution of the second order non-linear equation is oscillatory.

REFERENCE:


