INTERVAL CRITERIA FOR OSCILLATION OF SECOND ORDER NON-LINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Abstract

Oscillation criteria are established in this paper for the second order non-linear neutral delay differential equations $\left[w(t)\left(x(t)+m(t)x(t-\tau)\right)'\right]'+n(t)f\left(x(t-\delta)\right)=0$

Where τ *and* δ *are nonnegative constants,* $w,m,n\in C([t_0,\infty),R)$, *and* $f\in C(R,R)$.*These results are different from known one in the sense that they are based on the information only on a sequence of subintervals of* $[t_0,\infty)$

Keywords –*Neutral differential equations, Oscillation, Criteria, Delay differential equations*.

INTRODUCTION

Consider the second order neutral delay differential equation

$$\left[w(t)\left(x(t)+m(t)x(t-\tau)\right)'\right]'+n(t)f\left(x(t-\delta)\right)=0 \qquad \dots\dots[A]$$

Where $t\geq t_0$, τ and δ are nonnegative constants$w$,

$m,n\in C([t_0,\infty),R)$, and$f\in C(R,R)$. Let as assume the following as ……..[a]

$q(t)\geq 0, w(t)>0, {}^{\infty}\left(1/_{w(s)}\right)\,ds=\infty,$

$f(x)/_{x}\geq\gamma>0$ for$x\neq 0.$

THEOREM- 1

If (a) $n(t)\geq o, w(t)>0, {}^{\infty}\left(1/_{w(s)}\right)\,ds=\infty, {}^{f(x)}/_{x}\geq\gamma>0$ …..[a]
For$x\neq 0.$

Holds and $x(t)$ eventually positive solution of equation (A), then $z(t) \geq o,\ z'(t) \geq o,$

$\left(w(t)z'(t)\right)' \leq o$ on interval $[T_0, \infty)$

For some $T_0 \geq t_0$ sufficiently large.

Moreover,

(i). If $0 \leq m(t) \leq 1$, then

$$\left(w(t)z'(t)\right)' + \gamma n(t)[1 - m(t-\delta)]z(t-\delta) \leq 0 \qquad \ldots\ldots\ldots [1.1]$$

(ii). if $-1 < \alpha \leq m(t) \leq 0$ then

$$\left(w(t)z'(t)\right)' + \gamma n(t)z(t-\delta) \leq 0 \qquad \ldots\ldots [1.2]$$

PROOF:

Without loss of generality assume that $x(t) > 0$ for all $t \geq T_0 - \tau - \delta$. since $n(t) \geq 0$ equation (A) implies that $\left(w(t)z'(t)\right)' \leq 0$ and $\left(w(t)z'(t)\right)'$ is decreasing.

It follows that

$\lim_{t\to\infty} w(t)z'(t) = l$

Let as prove that
$w(t)z'(t) \geq 0$
And[1.1]and[1.2] holds.

(i) If $0 \leq m(t) \leq 1$, then prove that $w(t)z'(t) \geq 0$

Otherwise there exist $t_1 \geq T_0$ Such that $z'(t_1) < 0$.

From $\left(w(t)z'(t)\right)' \leq 0$. It follows that

$$z(t) \leq z(t_1) + w(t_1)z'(t_1)\int_{t_1}^{t} \frac{1}{r(s)}\, ds.$$

Hence by condition (a) we have $\lim_{t\to\infty} z(t) = -\infty$ which contradict that $z(t) > 0$ for $t \geq t_0$.

Now observe that from (A) we have

$$[w(t)z'(t)]' + n(t)f\left(x(t-\delta)\right) = 0. \qquad \ldots\ldots\ldots [1.3]$$

From the condition

$n(t) \geq o, w(t) > 0, {}^{\infty}\left({}^{1}/_{w(s)}\right) ds = \infty, {}^{f(x)}/_{x} \geq \gamma > 0$
for $x \neq 0$.

And by[1.3] we get

$$[w(t)z'(t)]' + \gamma n(t)[z(t-\delta) - m(t-\delta)x(t-\tau-\delta)] \leq 0$$

Which in view of the fact that $z(t) \geq x(t)$ and

If $z(t)$ is increasing we get

$$\left(w(t)z'(t)\right)' + \gamma n(t)[1 - m(t-\delta)]z(t-\delta) \leq 0$$

(ii) If $-1 < \alpha \le m(t) \le 0$ then prove that

$\lim_{t\to\infty} w(t)z'(t) = l \ge 0.$

Otherwise $l < 0$ then it we get $\lim_{t\to\infty} z(t) = -\infty$ which clime that $z(t)$ cannot be eventually negative on $[T_0, \infty)$

If that is the case it can consider two mutually exclusive cases.

CASE 1:

$x(t)$ is unbounded then there exist an increasing sequence $\{t_k\}, t_k \to \infty\ k \to \infty$

Such that

$x(t) = sup_{t\le t_k} x(t)$ and $x(t_k) \to \infty$ as $t_k \to \infty$

We find that $z(t_k) = x(t_k) + m(t_k)x(t_k - \tau) \ge x(t_k)(1 + m(t_k)) \ge 0$

Contradicts the fact that $\lim_{t\to\infty} z(t) = -\infty$

CASE 2:

$x(t)$ is bounded then there exist an sequence $\{t_k\}$ Such that $\lim_{k\to\infty} x(t_k) = \limsup_{t\to\infty} x(t).$

Since the sequence $\{x(t_k - \tau)\}$ and $\{m(t_k)\}$ and bounded, there exist convergent subsequences. Therefore, without loss of generality, we may suppose that $\lim_{k\to\infty} x(t_k - \tau)$ and $\lim_{k\to\infty} p(t_k)$ exist.

Hence,

$$0 > \lim_{k\to\infty} z(t_k) = \lim_{k\to\infty}[x(t_k) + m(t_k)x(t_k - \tau)]$$

$$\ge \lim_{k\to\infty}[x(t_k) + m(t_k)x(t_k)]$$

$$\ge \limsup_{t\to\infty} x(t)\left[1 + \lim_{k\to\infty} m(t_k)\right]$$

$$\ge 0.$$

This is also a contradiction.

Thus we must have $l \ge 0$, which implies that $z(t)$ must be eventually positive.

i.e.) there exist $t_* \ge t_0$ such that $z(t) > 0$ for all $t \ge t_*$. Otherwise since $\lim_{t\to\infty} w(t)z'(t) = l \ge 0$, and $w(t)z'(t)$ is nonincreasing, we must have $z(t) < 0$ for some $t \ge t_0$.

We therefore have $z(t) < 0, z'(t) \ge 0, (w(t)z'(t))' \le 0$ on $[T_0, \infty)$ for some $T_0 \ge t_0$ sufficiently large.

From condition (a), we have $f(x(t-\delta)) \ge \gamma x(t-\delta) \ge \gamma z(t-\delta)$ for $t \ge t_* + \delta$ sufficiently large. And we find equation (A) implies the equation for $z(t)$.

$0 = [w(t)z'(t)]' + n(t)f\big(x(t-\delta)\big) \geq [w(t)z'(t)]' + \gamma n(t)z(t-\delta)$
at $t \in [t_*, \infty)$.

Hence proved.

RESULT:

In theorem 1 if $x(t)$ is an eventually negative solution of (A), then the relevant result hold.

In the sequence we say that a function $H = H(t,s)$ belongs to function class K, denoted H∈K, if $H \in C(D, R_+ = (0,\infty))$ and $k \in C^1(D, R_+)$

Where $D = \{(t,s): -\infty < s < t < \infty\}$ which satisfies

$H(t,t) = 0, H(t,s) > 0,$ for $t > s,$ ……. (B1)

And has partial derivatives

$\frac{(\partial H(t,s))}{\partial t}$ and $\frac{(\partial H(t,s))}{\partial s}$ on D such that

$$\frac{\partial}{\partial s}(H(t,s)k(t)) = h_1(t,s)\sqrt{(H(t,s)k(t))},$$

$$\frac{\partial}{\partial s}(H(t,s)k(t)) = -h_2(t,s)\sqrt{(H(t,s)k(t))}, \qquad \text{… … … (B2)}$$

Where $h_1, h_2 \in C(D,R)$.

THEOREM- 2

If (a)

$n(t) \geq o, w(t) > 0, \quad {}^{\infty}\left({}^{1}/_{w(s)}\right) ds = \infty, {}^{f(x)}/_{x} \geq \gamma > 0$

For $x \neq 0$.holds and $x(t)$ be a solution of (A) such that $x(t) \neq 0$ $[T_0 - \tau - \delta, \infty)$ for some $T_0 \geq t_0$.For any $g \in C^1([t_0,\infty), R)$,
let

$$r(t) = -v(t)\left\{\frac{w(t)z'(t)}{z(t-\delta)} + w(t-\delta)g(t)\right\}, \qquad \text{…. . [2.1]}$$

Where $t \in [T_0, \infty)$. Then for any $H \in K$,

(i). If $0 \leq m(t) \leq 1$ and $t \in [c,b) \subset [T_0,\infty)$ then

$$\int_c^b H(b,s)\phi_1(s)ds \leq -H(b,c)k(c)w(c) + \frac{1}{4}\int_c^b w(s-\delta)v(s)h_2^2(t,s)ds.$$

(ii). If $-1 < \alpha \leq m(t) \leq 0$ and $t \in [c,b) \subset [T_0,\infty)$ then

$$\int_c^b H(b,s)\phi_2(s)ds \leq -H(b,c)k(c)r(c) + \frac{1}{4}\int_c^b w(s-\delta)v(s)h_2^2(t,s)ds.$$

PROOF

CASE (i)

Without loss of generality assume that $x(t) > 0$ for all
$t \geq T_0 - \tau - \delta$

Differentiating [2.1] and make use of (A) and by theorem (1.) Case: (i). we get that for $s \in [c, b)$

$$r'(s) = 2g(s)r(s) - v(s)\left\{\frac{(w(s)z'(s))'}{z(s-\delta)} - \frac{w(s)z'(s)z'(s-\delta)}{z^2(s-\delta)} + [w(s-\delta)g(s)]'\right\}$$

$$\geq 2g(s)r(s) + v(s)\left\{\gamma n(s)[1-m(s-\delta)] + \frac{w(s)z'(s)z'(s-\delta)}{z^2(s-\delta)} - [w(s-\delta)g(s)]'\right\}$$

From the fact that $w(s)z'(s)$ is decreasing, we get

$w(s)z'(s) \leq w(s-\delta)z'(s-\delta)$
For $s \geq T_0$

From the above we know that

$$r'(S) \geq 2g(s)r(s) + v(s)\left\{\gamma n(s)[1-m(s-\delta)] + \frac{1}{w(s-\delta)}\left(\frac{w(s)z'(s)}{z(s-\delta)}\right)^2 - [w(s-\delta)g(s)]'\right\}$$

$$= 2g(s)r(s) + v(s)[w(s-\delta)g(s)]' + v(s)\left\{\gamma n(s)[1-m(s-\delta)] + \frac{1}{w(s-\delta)}\left(\frac{w(s)}{v(s)} - w(s-\delta)g(s)\right)^2\right\}$$

$$= \phi_1(s) + \frac{1}{w(s-\delta)v(s)}r^2(s).$$

It follows that

$$\phi_1(s) \leq r'(s) - \frac{1}{w(s-\delta)v(s)}r^2(s). \qquad \text{……… [2.2]}$$

Multiplying [2.2] by $H(t,s)k(s)$,

And integrating it with respect to s from c to t for $t \in [c,b)$, and using the result of (B1) and (B2), one can get

$$\int_c^t H(t,s)k(s)\phi_1(s)\,ds \leq \int_c^t H(t,s)k(s)r'(s)\,ds - \int_c^t \frac{1}{w(s-\delta)v(s)}H(t,s)k(s)r'(s)\,ds$$

$$= -H(t,c)k(c)r(c) + \int_c^t h_2(t,s)\sqrt{H(t,s)k(s)}\,r(s)\,ds - \int_c^t \frac{H(t,s)k(s)}{w(s-\delta)v(s)}r^2(s)\,ds$$

$$= -H(t,c)k(c)r(c) - \int_c^t \left[\sqrt{\frac{H(t,s)k(s)}{w(s-\delta)v(s)}}\,r(s) - \frac{1}{2}\sqrt{w(s-\delta)v(s)}\,h_2(t,s)\right]^2 ds + \frac{1}{4}\int_c^t w(s-\delta)v(s)\,h_2^2(t,s)\,ds$$

$$\leq -H(t,c)k(c)r(c) + \frac{1}{4}\int_c^t w(s-\delta)v(s)\,h_2^2(t,s)\,ds.$$

Letting $t \to b^-$ in the above (i) is proved.

$$\int_c^b H(b,s)\phi_1(s)\,ds \le -H(b,c)k(c)r(c) + \frac{1}{4}\int_c^b w(s-\delta)v(s)\,h_2^2(t,s)\,ds.$$

CASE (ii)

Without loss of generality assume that $x(t) > 0$ for all $t \ge T_0 - \tau - \delta$

Differentiating[2.1] and make use of (A) and by theorem (1)

Case: (ii) we get that for $s \in [c,b)$

$$r'(S) = 2g(s)r(s) - v(s)\left\{\frac{\left(w(s)z'(s)\right)'}{z(s-\delta)} - \frac{w(s)z'(s)z'(s-\delta)}{z^2(s-\delta)} + [w(s-\delta)g(s)]'\right\}$$

$$\ge 2g(s)r(s) + v(s)\left\{\gamma n(s) + \frac{1}{w(s-\delta)}\frac{w(s)z'(s)w(s-\delta)z'(s-\delta)}{z^2(s-\delta)} - [w(s-\delta)g(s)]'\right\}$$

Similar to the proof of theorem (1) we can show the following inequality:

$$\phi_2(s) \le r'(s) - \frac{1}{w(s-\delta)v(s)}r^2(s). \qquad \dots \dots [2.3]$$

Multiplying [2.3] by $H(t,s)k(s)$,

And integrating it with respect to s from c to t for $t \in [c,b)$,

$$\int_c^t H(t,s)k(s)\phi_2(s)\,ds \le \int_c^t H(t,s)k(s)r'(s)\,ds - \int_c^t \frac{1}{w(s-\delta)v(s)}H(t,s)k(s)r^2(s)\,ds$$

$$= -H(t,c)k(c)r(c) + \int_c^t h_2(t,s)\sqrt{H(t,s)k(s)}\,r(s)\,ds - \int_c^t \frac{H(t,s)k(s)}{w(s-\delta)v(s)}r^2(s)\,ds$$

$$= -H(t,c)k(c)r(c) - \int_c^t \left[\sqrt{\frac{H(t,s)k(s)}{w(s-\delta)v(s)}}\,r(s) - \frac{1}{2}\sqrt{w(s-\delta)v(s)}\,h_2(t,s)\right]^2 ds + \frac{1}{4}\int_c^t w(s-\delta)v(s)\,h_2^2(t,s)\,ds$$

$$\le -H(t,c)k(c)r(c) + \frac{1}{4}\int_c^t w(s-\delta)v(s)\,h_2^2(t,s)\,ds.$$

Letting $t \to b^-$ in the above (ii) is proved.

$$\int_c^b H(b,s)\phi_2(s)\,ds \le -H(b,c)k(c)r(c) + \frac{1}{4}\int_c^b w(s-\delta)v(s)\,h_2^2(t,s)\,ds.$$

Hence proved.

CONCLUSION:

Throughout this work, we discussed some definition and theorems on Interval criteria for oscillation of second order non-linear neutral delay differential equations and then we discussed for the oscillation of second order non-linear neutral delay differential equations with non-negative constants on the interval $[t_0, \infty)$. finally we establish that, when the co-efficient of neutral delay differential equations is zero so that the solution of the second order non-linear equation is oscillatory

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