Integral Transforms and Ulam Stability for Differential Equations

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Abstract

In this article, we study to prove the Hyers – Ulam stability ($see\ denote\ \mathcal{HU}s$) of the differential equation using Fourier Transform.

Index Terms- $\mathcal{HU}s$, linear differential equation, Fourier Transform.

I. Introduction

The study of stability for a variety of functional equations is invented by a famous mathematician S. M.Ulam [23] in the year 1940. He raised the question concerning the stable of functional equations: "Give properties in order of linear function near an approximately linear function to exist". The first solution was brilliantly answered to the problem of Ulam for Cauchy additive functional equation based on Banach spaces by D. H. Hyers [24] in 1941. A generalized result to Ulam's question for approximately linear mappings was proposed by Th. M. Rassias [25] in 1978. During these days, a lot of mathematicians contributed to the expansion of the Ulam's problem with other functional equations with other spaces in different directions. [3-5, 8-11, 14-17, 28].

The differential equation

$$\zeta\left(g,\varphi,\varphi',\varphi'',...,\varphi^{(n)}\right)=0$$

is called $\mathcal{HU}s$ if for a have $\grave{o}>0$ and a function φ such that

$$\left|\zeta\left(g,\varphi,\varphi',\varphi'',...,\varphi^{(n)}\right)\right|\leq\grave{o},$$

there exists some φ_a of equation such that $|\varphi(s)-\varphi_a(s)|\leq H(\grave{o})$ and $\lim_{\grave{o}\to 0}H(\grave{o})=0.$

Oblaza seems to be the first researcher study the $\mathcal{HU}s$ of differential equations [12, 13]. Thereafter [18], which exist the $\mathcal{HU}s$ of the linear differential equation $\psi'(1)=\lambda\psi(1)$. S. M. Jung continuously posed the general setting for $\mathcal{HU}s$ of first order differential equations in [30-32].

The matrix method and $\mathcal{HU}s$ of differential equations with a coefficient posed by S. M. Jung [33] in 2006. The Generalized $\mathcal{HU}s$ of higher order linear differential equation use to Laplace transform proved by Alqifiary and Jung in 2014.

In recent years, J.M. Rassias *et al.*[22] proved the Mittag-Leffler- $\mathcal{HU}s$ of first order and second order differential equations use to Fourier transfor. Then recently, J.M. Rassias [1] investigated the Mittag-Leffler-HUs of first order and second order nonlinear initial value problems Applying the Laplace transform method. HUs of differential equations is now being studied [2, 6, 7, 19-21, 26, 27, 29] and the investigation is ongoing.

Our main aim of this manuscript, we study the $\mathcal{HU}s$ of the differential equation

$$\varphi'(r) + l\ \varphi(r) = 0 \tag{1}$$

and the non-homogeneous differential equation

$$\varphi'(r) + l\ \varphi(r) = c(r) \tag{2}$$

here l is a scalar, $\varphi(r)$ and $c(r)$ are the continuously differentiable function.

II. Preliminaries

Here, we recall the notations, definitions and theorems, it will be very useful to prove our main results.

The function $u : (0, \infty) \to \mathbb{F}$ of exponential order if there exists a constants $M(> 0) \in R$ such that $|u(t)| \leq Z$ at for each $r > 0$. For each function $u : (0, \infty) \to \mathbb{F}$ of exponential order, we define the Fourier Transform of u by

$$U(s) = \int_0^\infty u(r) e^{-ist} dr.$$

and

$$u(r) = \frac{1}{2\pi i} \lim_{n\to\infty} \int_{\alpha - iR}^{\alpha + iR} U(s) e^{isr} dr.$$

is says the inverse Fourier transforms.

Definition 2.1. (Convolution) Let two functions u and v, is Lebesgue integral on $(-\infty, +\infty)$. If S represent the set of φ is Lebesgue integral

$$w(\alpha) = \int_{-\infty}^{\infty} u(r)\, v(\varphi - r)\, dr$$

exists. This integral choose a function w on S is says the convolution of u and v. We also write $w = u * v$ to represent this function.

Definition 2.3. The Eq.(1.1) has $\mathcal{HUs}$, if there exists a constant $Z > 0$ with the following property: For each$\varepsilon > 0$, let $\varphi(r)$ be a continuously differentiable function satisfies the inequality $|\varphi'(r) + l\ \varphi(r)| \leq \varepsilon$, then there exists some $\psi(r)$ satisfies the Eq.(1.1) such that $|\varphi(r) - \psi(r)| \leq Z\varepsilon$, for each $r > 0$. Hence such Z as the $\mathcal{HUs}$ constant for the Eq.(1.1).

Definition 2.4. The Eq.(1.2) has $\mathcal{HUs}$, if there exists a constant $Z > 0$ with the following property: For each$\varepsilon > 0$, let $\alpha(r)$ be a continuously differentiable function satisfies the inequality $|\varphi'(r) + l\ \varphi(r) - c(r)| \leq \varepsilon$, then there exists some $\psi(r)$ satisfies the Eq.(1.2) such that $|\varphi(r) - \psi(r)| \leq Z\varepsilon$, for each $r > 0$. Hence such Z as the $\mathcal{HUs}$ constant for the Eq.(1.2).

III. Main Results

In the following theorems, we prove the $\mathcal{HUs}$ of the homogeneous and Eq.(1.1), (1.2). Firstly, we prove the $\mathcal{HUs}$ of first order Eq.(1.1).

Theorem 3.1. *The Eq.*(1.1) *has* $\mathcal{HUs}$.

Proof.

Let l be a constant in $\mathfrak{F}$. For each$\varepsilon > 0$, there exists a positive constant Z such that $\varphi(r)$ be a continuously differentiable function satisfies the inequality

$$|\varphi'(r) + l\ \varphi(r)| \leq \varepsilon, \tag{3.1}$$

for each $r > 0$. We will prove that, there exists a solution $\psi(r)$ satisfying the $\psi'(r) + l\,\psi(r) = 0$ such that

$$|\varphi(r) - \psi(r)| \leq Z\varepsilon$$

for each $r > 0$.

Let us define a function $a(r)$ such that

$$a(r) =: \varphi'(r) + l\ \varphi(r)$$

for each $r > 0$. In view of (3.1), we have $|a(r)| \leq \varepsilon$. Now, taking Fourier transform to $a(r)$, we have

$$\mathcal{F}\{a(r)\} = \mathcal{F}\{\varphi'(r) + l\ \varphi(r)\}$$
$$A(\xi) = \mathcal{F}\{\varphi'(r)\} + l\,\mathcal{F}\{\varphi(r)\} = -i\xi\Phi(\xi) + l\,\Phi(\xi) = (l - i\xi)\Phi(\xi)$$
$$\Phi(\xi) = \frac{A(\xi)}{(l - i\xi)}$$

Thus

$$\mathcal{F}\{\varphi(r)\} = \Phi(\xi) = \frac{A(\xi)\,(l + i\xi)}{l^2 - \xi^2} \tag{3.2}$$

Taking $B(\xi) = \frac{1}{(l - i\xi)}$, then we have

$$\mathcal{F}\{b(r)\} = \frac{1}{(l - i\xi)} = b(r) = \mathcal{F}^{-1}\left\{\frac{1}{(l - i\xi)}\right\}.$$

Now, we set $\psi(r) = e^{-lr}$ and taking Fourier transform on both sides, we get

$$\mathcal{F}\{\psi(r)\} = \Psi(\xi) = \int_{-\infty}^{\infty} e^{-lr}\, e^{isr}\, dr = 0 \tag{3.3}$$

Now,

$$\mathcal{F}\{\psi'(r) + l\,\psi(r)\} = \mathcal{F}\{\psi'(r)\} + l\,\mathcal{F}\{\psi(r)\} = -i\xi\Psi(\xi) + l\,\Psi(\xi)$$
$$= (l - i\xi)\Psi(\xi).$$

Applying (3.3), we have $\mathcal{F}\{\psi'(r) + l\,\psi(r)\} = 0$, $\mathcal{F}$ is one-to-one operator, thus $\psi'(r) + l\,\psi(r) = 0$, Hence $\psi(r)$ is a solution of the Eq. (1.1). Plucking (3.2) and (3.3) we can obtain $\mathcal{F}\{\varphi(r)\} - \mathcal{F}\{\psi(r)\} = \Phi X(\xi) - \Psi(\xi) = \frac{P(\xi)\,(l + i\xi)}{l^2 - \xi^2} = A(\xi)\,B(\xi) =$ $\mathcal{F}\{a(r)\}\,\mathcal{F}\{b(r)\}$

$$\Rightarrow \mathcal{F}\{\varphi(r) - \psi(r)\} = \mathcal{F}\{a(r) * b(r)\}.$$

Since the operator $\mathcal{F}$ is one-to-one and linear, which gives $\varphi(r) - \psi(r) = a(r) * b(r)$.
Applying modulus on both sides, we have

$$|\varphi(r) - \psi(r)| = |a(r) * b(r)| = \int_{-\infty}^{\infty} a(r)\,b(r - s)\,ds$$
$$\leq |a(r)| \left| \int_{-\infty}^{\infty} b(r - s)ds \right| \leq Z\varepsilon$$

Here $Z = \left|\int_{-\infty}^{\infty} b(r - s)ds\right|$, the integral exists for each value of t. Hence, by the virtue of Definition 2.3 the Eq.(1.1) has the $\mathcal{HUs}$.

Theorem 3.2. *The Eq.*(1.2) *has* $\mathcal{HUs}$.

PROOF.

Let l be a constant in $\mathfrak{F}$. For each $\varepsilon > 0$, then there exists a non-negative constant K such that $\varphi(r)$ be a continuously differentiable function satisfies the inequality

$$|\varphi'(r) + l\,\varphi(r) - c(r)| \leq \varepsilon, \tag{3.4}$$

for each $r > 0$. We prove that, there exists some solution $\psi(r)$ satisfying the differential equation $\psi'(r) + l\,\psi(r) - c(r) = 0$ such that

$$|\varphi(r) - \psi(r)| \leq Z\varepsilon$$

for each $r > 0$.
Let us define a function $p(r)$ such that

$$a(r) =: \varphi'(r) + l\,\varphi(r) - c(r))$$

for each $r > 0$. In see the Eq.(3.4), we have $|p(r)| \leq Z\varepsilon$. Now, applying Fourier transform to $c(r)$, we obtain

$$\mathcal{F}\{a(r)\} = \mathcal{F}\{\varphi'(r) + l\,\varphi(r) - c(r)\}$$
$$A(\xi) = \mathcal{F}\{\varphi'(r)\} + l\,\mathcal{F}\{\varphi(r)\} - \mathcal{F}\{c(r)\} = -i\xi\Phi(\xi) + l\,\Phi(\xi) - C(\xi)$$
$$= (l - i\xi)\Phi(\xi) - C(\xi)$$
$$\Phi(\xi) = \frac{A(\xi) - C(\xi)}{(l - i\xi)}$$

Thus

$$\mathcal{F}\{\varphi(r)\} = \Phi(\xi) = \frac{A(\xi)(l + i\xi) - C(\xi)}{l^2 - \xi^2} \tag{3.5}$$

Taking $B(\xi) = \frac{1}{(l - i\xi)}$, then we have

$$\mathcal{F}\{b(r)\} = \frac{1}{(l - i\xi)} = b(r) = \mathcal{F}^{-1}\left\{\frac{1}{(l - i\xi)}\right\}.$$

Now, set $\psi(r) = e^{-lr}$ and applying Fourier transform on both sides, we obtain

$$\mathcal{F}\{\psi(r)\} = \Psi(\xi) = \int_{-\infty}^{\infty} e^{-lr} e^{isr} dr = 0 \qquad (3.6)$$

Now,

$$\mathcal{F}\{\psi'(r) + l\,\psi(r) - c(r)\} = \mathcal{F}\{\psi'(r)\} + l\,\mathcal{F}\{\psi(r)\} - \mathcal{F}\{c(r)\}$$
$$= -i\xi\Psi(\xi) + l\,\Psi(\xi) - \mathrm{C}(\xi) = (l - i\xi)\Psi(\xi) - \mathrm{C}(\xi).$$

Using (3.3), we have $\mathcal{F}\{\psi'(r) + l\,\psi(r) - c(r)\} = 0$, since $\mathcal{F}$ is one-to-one operator, thus $\psi'(r) + l\,\psi(r) - c(r) = 0$, Hence $\psi(r)$ is a solution of the Eq.(1.1). Plucking (3.5) and (3.6) we can obtain

$$\mathcal{F}\{\varphi(r)\} - \mathcal{F}\{\psi(r)\} = \Phi(\xi) - \Psi(\xi) = \frac{A(\xi)\,(l + i\xi) - \mathrm{C}(\xi)}{l^2 - \xi^2} = A(\xi)\,\mathrm{B}(\xi) = \mathcal{F}\{a(r)\}\,\mathcal{F}\{\mathrm{b}(r)\}$$

$$\Rightarrow \mathcal{F}\{\varphi(r) - \psi(r)\} = \mathcal{F}\{a(r) * b(r)\}.$$

Since the operator $\mathcal{F}$ is one-to-one and linear, which gives $\varphi(r) - \psi(r) = a(r) * b(r)$. Applying modulus on both sides, we have

$$|\varphi(r) - \psi(r)| = |a(r) * b(r)| = \int_{-\infty}^{\infty} \mathrm{a}(r)\, b(r - s)\, ds$$

$$\leq |a(r)| \left| \int_{-\infty}^{\infty} b(r - s) ds \right| \leq Z\varepsilon.$$

Here $Z = \left|\int_{-\infty}^{\infty} b(r - s) ds\right|$, the integral exists for each value of t. Therefore, by the virtue of Definition 2.4 the Eq. (1.2) has $\mathcal{HUs}$.

IV. Conclusion

We are prove that one of the $\mathcal{HUs}$, namely the $\mathcal{HUs}$ of a of the differential equations of first order with constant coefficients using the Fourier Transforms method. That is, we solve the sufficient criteria for $\mathcal{HUs}$ of the differential equation of first order using Fourier Transforms method.

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