Itration methods for non - linear equations

Resercher : pohanmal Sayed naqibullah monsef

teacher of mathematics depatement

Abstract

In this paper, we suggest and analyze a class of new iterative method for solving nonlinear equations using a new decomposition technique. We show that three-step iterative method has fourth-order convergence. Several numerical examples are given to illustrate the etticiency and performance of the proposed three-step iterative method. Our results can be considered as an alternative to the recent three-step iterative system.

Keywords*: Nonlinear Equations; Systems of Nonlinear Equations; Iterative Methods*

1.interduction

In recent years, much attention has been given to develop several iterative methods for solving nonlinear equations, see [1–6]. These methods can be classified as one-step and two-step methods. Two-step methods have been suggested by combining the well-known Newton method with other one-step methods. Abbasbandy [1] and Chun [3] has proposed and studied several one-step and two-step iterative methods with higher-order convergence by using the decomposition technique of Adomian [2]. In their methods, they have to used the higher order differential derivatives which is a serious drawback. To overcome this draw back, Noor and Noor [6] used another decomposition technique to suggest and analyze three-step iterative method for solving nonlinear equations. In this paper, we suggest and analyze a new family of multi-step methods for solving nonlinear equations using a different type of decomposition. This decomposition technique is mainly due to Noor [5].

Our method is very simple as compared with Adomian decomposition method. In particular, we introduce a new three-step iterative method for solving nonlinear equations. We have shown that the convergence of three-step method only involve first derivative of the functions. We also consider the convergence of the proposed method. Several numerical examples are given to illustrate the efficiency and the performance of the new iterative methods. Our results can be considered as an important improvement and refinement of the previously known results.

2. Iterative methods and convergence analysis

Consider the nonlinear equation

$$f(x) = 0 \qquad (1)$$

We assume that α is a simple root of (1) and β is an initial guess sfficiently close to α .

By using the Taylor's series, we have

$$f(\beta) + f'(\beta)(x-\beta) + \frac{(x-\beta)^2}{2} f''(\beta) \qquad (2)$$

where c is the initial approximation for a zero of (1).

We can rewrite Eq (2) in the following form

$$x = c + N(x) \tag{3}$$

where

$$x = \beta - \frac{f(\beta)}{f'(\beta)} \tag{4}$$

And

$$N(x) = -\frac{(x-\beta)}{2f'(\beta)} f''(\beta) \tag{5}$$

We now construct a sequence of high-order iterative methods by using the following decomposition method, which is mainly due to Noor [5].

This decomposition of the nonlinear operator N(x) is quite different than that of Adomian decomposition.

The main idea of this technique is to look for a solution of Eq. (3) having the series form:

$$x = \sum_{i=0}^{\infty} x_i \tag{6}$$

The nonlinear operator N can be decomposed as

$$N\left(\sum_{i=0}^{\infty} x_i\right) = N(x_0) + \sum_{i=0}^{\infty}\left\{N\left(\sum_{j=0}^{i} x_i\right)\right\} \tag{7}$$

Combining (3), (6) and (7), we have

$$\sum_{i=0}^{\infty} x_i = C + N(x_0) + \sum_{i=0}^{\infty}\left\{N\left(\sum_{j=0}^{i} x_j\right)\right\}. \tag{8}$$

Thus we have the following terative schemes:

$$\begin{aligned} &x_0 = c, \\ &x_1 = N(x_0), \\ &x_2 = N(x_0 + x_1), \\ &. \\ &. \\ &. \\ &x_{n+1} = N(x_0 + x_1 + \ldots + x_m), \quad n = 1,2,3 \\ &\textit{then} \\ &x_1 + x_2 + \ldots + x_{n+1} = N(x_0) + \ldots + N(x_0 + x_1 + \ldots + x_m), \ n = 1,2,3\ldots \end{aligned} \tag{10}$$

And

$$x = c + \sum_{i=0}^{\infty} x_i \tag{11}$$

If follows from (3), (4), (10) and (11) that

$$x_0 = c = \beta - \frac{f(\beta)}{f'(\beta)} \tag{12}$$

And

$$x_1 = N(x_0) = -\frac{(x-\beta)}{2f'(\beta)} f''(\beta) \tag{13}$$

From (11) and (12), we have

$$x = c = x_0 = \beta - \frac{f(\beta)}{f'(\beta)}$$

This allows us to suggest the following one-step iterative method for solving the nonlinear equation (1).

Algorithm 2.1. For a given x_0, compute the approximate solution x_{n+1} by the iterative scheme.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \quad n = 0,1,2...$$

which is known as Newton method and Algorithm 2.1 has a second-order convergence, see [5] and the refera ences therein.

Again using (10), (12) and (13), we conclude that

$$x = c + x_1 = x_0 + N(x_0), \tag{14}$$

$$= \beta - \frac{f(\beta)}{f'(\beta)} - \frac{(x_0-\beta)^2}{2f'(\beta)} f''(\beta), \tag{15}$$

Using this relation, we can suggest the following two-step iterative method for solving Ex. (1) as

Algorithm 2.2. For a given x_0, compute the approximate solution x_{n+1} by the iterative schemes:

Predictor-step:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0,$$

Corrector-step:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{(y_n - x_n)^2}{2f'(\beta)} f''(\beta), \quad n = 0,1,2$$

Algorithm 2.2 can be written in the following form:

Algorithm 2.3. For a given x_0, compute the approximation solution x_{n+1} be iterative method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)}{2f'^3(x_n)} f''(x_n), \quad n = 0,1,2$$

which is known as the Householder's iteration, see [1] and was obtained by Abbasbandy [1] using the Adou mian decomposition, whereas our method is very simple and natural one. For the convergence analysis of Algorithm 2.3, see [3,5].

Again using (11) and (7), we can calculate

$$x_2 = N(x_0 + x_1) = -\frac{f(x_0 + x_1 - \beta)^2}{2f'(\beta)} f''(\beta) \tag{16}$$

From (11), (13), (14) and 916) we conclude that

$$\begin{aligned} x &= c + x_1 + x_2 = x_0 + N(x_0) + N(x_0 + x_1) \\ &= \beta - \frac{f(\beta)}{f'(\beta)} - \frac{(x_0 - \beta)^2}{f'(\beta)} f''(\beta) - \frac{(x_0 - x_1 - \beta)^2}{2f'(\beta)} f''(\beta). \end{aligned}$$

Using this, we can suggest and analyze the following three-step iterative method for solving nonlinear equaU tion (1) and this is the main motivation of this paper.

Algorithm 2.4. For a given x_0, compute the approximate solution x_{n+1} by the iterative schemes:

Predictor -steps:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \tag{17}$$

$$z_n = -\frac{(x_0 - x_n)^2}{2f'(x_n)} f''(x_n) \tag{18}$$

Corrector-step:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{(y_n - x_n)^2}{2f'(x_n)} f''(x_n) - -\frac{(y_n - z_n - x_n)^2}{2f'(x_n)} f''(x_n), \quad n = 0,1,2... \tag{19}$$

Algorithm 2.4 is called the three- step iterative method for solving non-linear equation (1).
We now study the convergence analysis of Algorithm 2.4 and this is the main motivation of our next result.

Theorem 2.1. Let $r \in I$ be a simple zero of sufficiently differentiable function $f: I \subseteq R \to R$ for an open interval I. If x_0 is sufficintly close to r, then the three-step iterative method defined by Algorithm 2.3 has fourth-order convergence.

Proof. Let r be a simple zero of f. Since f is sufficiently differentiable, by expanding $f(x_n)$ and $f'(x_n)$ about r, we get

$$f(x_n) = f'(r)\left[e_n + c_2 e_n^2 + c_3 e_n^3 + ...\right], \tag{20}$$

$$f'(x_n) = f'(r)\left[1 + 2c_2 e_n + 3c_3 e_n^2 + 4e_n^3..\right], \tag{21}$$

Where $c_k = \frac{1}{k!}\frac{f^{(k)}(r)}{f'(r)}, \quad k = 1,2,3,....and\ e_n = x_n - r$. Now from (20) and (21) we have

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + ..., \tag{22}$$

From (17) and (220, we get

$$y_n = r + c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + ..., \tag{23}$$

Now expanding $f(y)$ in place of r and using (23), we have

$$f(y_n) = f'(r)\left[c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + ...\right], \tag{24}$$

Now from (21) and (24), we get

$$z_n = -c_2 e_n^2 - 2\left(c_3 - c_2^2\right) e_n^3 + \dots \tag{25}$$

Now again expanding $f\left(y_n + z_n\right)$ in place of r and using (23) and (25), we have

$$f\left(y_n + z_n\right) = f'(r)\left[2c_2^2 e_n^3 + \dots\right] \tag{26}$$

From (25) and (26), we get

$$\frac{f\left(y_n + z_n\right)}{f'\left(x_n\right)} = 2c_2^2 e_n^3 - 2c_2 e_n^4 + \dots.. \tag{27}$$

From (22), (25) and (27), it follows that algorithm 2.4 has forth order convergence.

3. Numerical example

We present some examples to illustrate the efficiency of the newly development three-step iterative methods in the paper. We compute the Newton method (NM), the method of

Table3.1

Some Example and comparison

	IT	x_n	$f\left(x_n\right)$	δ
$f_1, x_0 = 2$				
NM	6	0.257530285439860760455367304940	2.93e-55	**9.1e-28**
AM	5	0.257530285439860760455367304940	1.0e-63	**1.45e-26**
HM	5	0.257530285439860760455367304940	0	**9.33e-43**
CM	4	0.257530285439860760455367304940	1.0e-63	**9.46e-29**
NR	4	0.257530285439860760455367304940	9.3e-28	**2.5e-28**
$f_2, x_0 = -1$				
NM	7	1.404491648215341226035086817800	-1.04e-50	**7.33e-26**
AM	5	1.404491648215341226035086817800	-5.81e-55	**1.39e-18**
HM	4	1.404491648215341226035086817800	-5.4e-62	**7.92e-21**
CM	5	1.404491648215341226035086817800	-2.0e-63	**1.31e-17**
NR	5	1.404491648215341226035086817800	-1.3e-40	**5.4e-41**
$f_3, x_0 = 1.5$				
NM	5	2.154434690031883721759293566500	2.06e-54	**5.64e-28**
AM	4	2.154434690031883721759293566500	-5.0e-63	**1.18e-25**
HM	4	2.154434690031883721759293566500	-5.0e-63	**9.8e-23**
CM	4	2.154434690031883721759293566500	-5.0e-63	**1.57e-22**
NR	4	2.154434690031883721759293566500	8.1e-45	**5.8e-46**
$f_4, x_0 = 1.7$				
NM	5	0.739085133215160641655312087670	-2.03e-32	**2.34e-16**

AM	4	0.739085133215160641655312087 67	-7.14e-47	**8.6e-16**
HM	4	0.739085133215160641655312087 67	-5.02e-59	**9.64e-20**
CM	4	0.739085133215160641655312087 67	0	**1.86e-53**
NR	4	0.739085133215160641655312087 67	-3.7e-54	**2.4e-54**
$f_5, x_0 = 3.5$				
NM	8	2	2.06e-42	**8.28e-22**
AM	5	2	0	**4.3e-22**
HM	5	2	0	**1.46e-24**
CM	5	2	0	**2.74e-24**
NR	5	2	1.75e-24	**5.8e-24**
$f_6, x_0 = -2$				
NM	9	-1.207647827130918927009416758 4	-2.27e-40	**2.73e-21**
AM	6	-1.207647827130918927009416758 4	-4.0e-63	**4.35e-45**
HM	6	-1.207647827130918927009416758 4	-4.0e-63	**2.57e-32**
CM	6	-1.207647827130918927009416758 4	-4.0e-63	**2.15e-36**
NR	6	-1.207647827130918927009416758 4	-1.0e-37	**4.9e-39**
$f_7, x_0 = 3.5$				
NM	13	3	1.52e-47	**4.2e-25**
AM	7	3	-4.33e-48	**2.25e-17**
HM	8	3	2.0e-62	**2.43e-33**
CM	8	3	2.0e-62	**2.12e-23**
NR	8	3	2.0e-62	**3.8e-26**

Homeier (HM), the method of Chun (CM) and the method (NR), introduced in this present paper. We used $\varepsilon = 10^{-15}$ (see Table 3.1).

The following stopping criteria is used for computer programs:

$(i)\ \ |x_{n+1} - x_n| < \varepsilon$

$(ii)\ \ |f(x_{n+1})| < \varepsilon$

The examples are the same as in Chun [3].

$$f_1(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5$$
$$f_2(x) = \sin^2 x - x^2 + 1,$$
$$f_3(x) = x^3 + 10,$$
$$f_4(x) = \cos x - x,$$
$$f_5(x) = (x-1)^3 - 1,$$
$$f_6(x) = x^2 - e^x - 3x + 2,$$
$$f_7(x) = e^{x^2+7x-30} - 1,$$

As for the convergence criteria, it was required that the distance of two consecutive approximations d for the zero was less than 10^{-15}. Also displayed are the number of iterations to approximate the zero (IT), the approx. imate zero x_* and the value $f\left(x_*\right)$.

4. Conclusion

In this paper, we have suggested a family of one-step, two-step and three-step iterative methods for solving nonlinear equations by using a different decomposition technique. Our methods of derivation of the iterative methods is very simple as compared with the Adomian decomposition methods. It is very important to note that the implementation of these multi-step methods does not require the computation of that higher-order derivatives compared to most other methods of the same order. This is another aspect of the simplicity of these methods.

Using the technique and idea of this paper, one can suggest and analyze higher-order multi-step iterative methods for solving nonlinear equations as well as system of nonlinear equations. This is the topic of further research.

References

[1] S. Abbasbandy, Improving Newton–Raphson method for nonlinear equations by modified Adomian decomposition method, Appl. Math. Comput. 145 (2003) 887–893.

[2] G. Adomian, Nonlinear Stochastic Systems and Applications to Physics, Kluwer Academic Publishers, Dordrecht, 1989.

[3] C. Chun, Iterative methods improving Newton's method by the decomposition method, Comput. Math. Appl. 50 (2005) 1559–1568.

[4] H.H. Homeier, On Newton-type methods with cubic convergenceJ.Comput. Appl. Math. 176 (2005) 425–432.

[5] M. Aslam Noor, Numerical Analysis and Optimization, Lecture Notes, COMSATS Institute of Information Technology, Islamabad, 2006.

[6] M. Aslam Noor, K. Inayat Noor, Three-step iterative methods for nonlinear equations, Appl. Math. Comput., in press, doi:10.1016/ j.amc.2006.05.055.