LEGENDRE'S FUNCTION OF THE FIRST AND SECOND KIND AND ITS GENERATING FUNCTION

Abdul Waris Qadirzada

Assistant Professor, Deportment of Mathematics, Samangan University, Samangan, Afghanistan
student, Deportment of Mathematics, Osmania University,Hyderabad Telangana, India

ABSTRACT

Legendre's polynomial is an important part of differential equation which divide in to two parts, they are first kind and second kind, these concepts explain by Legendre's equation. Here we will discuss about Legendre's equation, Legendre's polynomial and generating function for Legendre's polynomials.

Keyword: *Legendre's equation, Legendre's polynomials, generating function, orthogonal property.*

---

1. Introduction

The Legendre differential equation arises in problems such as the flow of an ideal fluid past a sphere, the determination of the electric field due to a charged sphere, and the determination of the temperature distribution in a sphere given its surface temperature.

Here we explained Legendre's equation in second section, in third section, polynomial of Legendre illustrated and in fourth section, we discussed about generating function and orthogonal property of Legendre polynomial [2].

2. Legendre's equation

The differential equqtion of the form $(1-x^2)y'' - 2xy' + n(n+1)y = 0 \cdots\cdots\cdots (1)$ is called Legendre's differential or simply Legendre's equation, where n is a constant.[3].

We now solve (1) in series of descending power of x. Let the series solution of (1) be

$$y = \sum_{m=0}^{\infty} c_m x^{k-m}, \qquad \text{where} \qquad c \neq 0 \cdots\cdots\cdots (2)$$

Differentiating (2) and then putting the values of y, y' and y'' into (1), we have

$$(1-x^2)\sum_{m=0}^{\infty} c_m (k-m)(k-m-1)x^{k-m-2} - 2x\sum_{m=0}^{\infty} c_m (k-m)x^{k-m-1} + n(n+1)\sum_{m=0}^{\infty} c_m x^{k-m} = 0 \quad (or)$$

$$\sum_{m=0}^{\infty} c_m (k-m)(k-m-1)x^{k-m-2} - \sum_{m=0}^{\infty} c_m \{(k-m)(k-m-1) + 2(k-m) - n(n+1)\}x^{k-m} = 0 \cdots\cdots (3)$$

Now, $(k-m)(k-m-1) + 2(k-m) - n(n+1)$

$= (k-m)^2 - (k-m) + 2(k-m) - n(n+1) = (k-m)^2 - n^2 + (k-m) - n$

$= (k-m+n)(k-m-n) + (k-m-n) = (k-m-n)(k-m+n+1).$

Hence (3) may be re-write as

$$\sum_{m=0}^{\infty} c_m (k-m)(k-m-1)x^{k-m-2} - \sum_{m=0}^{\infty} c_m (k-m-n)(k-m+n+1)x^{k-m} = 0 \cdots\cdots\cdots (4)$$

(4) is an identity. To get the indicial equation, we equate to zero the coefficient of the highest power of x, namely x^k in (4) and optain $c_0(k-m)(k+n+1)=0$ (or) $(k-m)(k+n+1)=0$ as $c_0 \neq 0 \cdots\cdots\cdots (5)$

So the root of (5) are $k=n,-(n+1)$. They are unequal and differ by an integer. The next lower power of x is $k-1$. So we equate to zero the coefficient of x^{k-1} in (4) and obtain

$$c_1(k-1-n)(k+n)=0 \cdots\cdots\cdots (6)$$

For $k=n$ and $-(n+1)$, neither $(k-1-n)$ nor $(k+n)$ is zero. So from (6), $c_1=0$. Finally, equating to zero the coefficient of x^{k-m} in (4), we have

$$c_{m-2}(k-m+2)(k-m+1)-c_m(k-m-n)(k-m+n+1)=0$$

$$c_m=\frac{(k-m+2)(k-m+1)}{(k-m-n)(k-m+n+1)}c_{m-2}.\cdots\cdots\cdots (7)$$

Putting $m=3,5,7,\ldots$ in (7) and noting that $c_1=0$, we have

$$c_1=c_3=c_5=c_7=\cdots=0.\cdots\cdots\cdots (8)$$

wich hold good for both $k=n$ and $k=-(n+1)$.

To obtain $c_2,c_4,c_6,\ldots$ etc, we consider two cases

CaseI. When $k=n$. Then, (7) become $c_m=-\frac{(n-m+2)(n-m+1)}{m(2n-m+)}c_{m-2}\ldots\ldots\ldots(9)$

Putting $m=2,4,6,\ldots$ in (9), we have

$$c_2=-\frac{n(n-1)}{2(2n-1)}c_0, \qquad c_4=-\frac{(n-2)(n-3)}{4(2n-3)}c_2=-\frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-1)(2n-3)}c_0$$

And so on. Re-write (2), we have for $k=n$

$$y=c_0x^n+c_1x^{n-1}+c_2x^{n-2}+c_3x^{n-3}+c_4x^{n-4}+\cdots \qquad \ldots\ldots\ldots(10)$$

Using (8) and the above values of $c_2,c_4,c_6,\ldots$etc., (10) becomes (after replacing c_0 by a)

$$y=a\left[x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-1)(2n-3)}x^{n-4}-\cdots\right].\ldots\ldots\ldots(11)$$

CaseII. When $k=-(n+1)$. Then, (7) becomes $c_m=\frac{(n+m-1)(n+m)}{m(2n+m+1)}c_{m-2}\ldots\ldots\ldots(12)$

Putting $m=2,\ 4,6,\ldots$ in (12), we have

$$c_2=-\frac{(n+1)(n+2)}{2(2n+3)}c_0, \qquad c_4=-\frac{(n+3)(n+4)}{4(2n+5)}c_2=-\frac{n(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\cdot(2n+3)(2n+5)}c_0$$

and so on. For $k=-(n+1)$, (2) gives

$$y=c_0x^{-n-1}+c_1x^{-n-2}+c_2x^{-n-3}+c_3x^{-n-4}+c_4x^{-n-5}+\cdots \qquad \ldots\ldots\ldots(13)$$

Using (8) and above values of $c_2,c_4,c_6,\ldots$ etc., (13) becomes

$$y=b\left[x^{-n-1}-\frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3}+\frac{n(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\cdot(2n+3)(2n+5)}x^{-n-5}+\cdots\right].\ldots\ldots\ldots(14)$$

Thus, two independent solution of of (1) are given by (11) and (14). If we take $a=[1\cdot 3\cdot 5\cdots\frac{2n-1}{n!}$, the solution (11) is denoted by $p_n(x)$ and is called Legendre's function of the first kind or Legendre's polynomial of degree n. Notice that (11) is terminating series and so it gives rise to a polynomial of degree n. Thus $p_n(x)$ is a solution of (1), Again, if we take $b=\frac{n}{[1\cdot 3\cdot 5\cdots(2n+1)]}$ the solution (14) is denoted by $Q_n(x)$ and is called Legendre's function of the second kind. Since n is positive integer, (14) is an infinite or non-terminating series and hence $Q_n(x)$ is not a polynomial. Thus $p_n(x)$ and $Q_n(x)$ are two linearly independent solution of (1). Hence the general solution of (1) is $y=Ap_n(x)+BQ_n(x)$, where A and B are arbitrary constants. $.\ldots\ldots\ldots(15)$

Remark 1. When there is no confusing regarding the variable x, we shall use a shorter notation P_n for $p_n(x)$ and p_n' for $\frac{dp_n(x)}{dx}$, Q_n for $Q_n(x)$ and Q_n' for $\frac{dQ_n(x)}{dx}$ etc

2.1 Another form of Legendre's polynomial $p_n(x)$

Legender's polynomial of degree n is denoted and define by

$p_n(x) = \frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot4\cdot(2n-1)(2n-3)}x^{n-4} - \cdots\right]. \ldots\ldots\ldots(1)$

We now re-write (1) in a compact form. The general term of polynomial (1) is given by

$\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{n!}\cdot(-1)^r \quad \frac{n(n-1)\ldots(n-2r+1)}{2\cdot4\cdot2r(2n-1)(2n-3)\ldots(2n-2r+1)}x^{n-2r} \ldots\ldots\ldots(2)$

Now, $1\cdot3\cdot5\cdot\ldots\cdot(2n-1) = \frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)\cdot2n}{2\cdot4\cdot6\cdot\ldots\cdot2n} = \frac{(2n)!}{(2\cdot1)(2\cdot2)(2\cdot3)\ldots(2\cdot n)} = \frac{(2n)!}{2^n\cdot1\cdot2\cdot3\ \ldots\ n} = \frac{(2n)!}{2^n\cdot n!}\ldots\ldots\ldots(3)$

Also, $n(n-1)\ldots.(n-2r+1) = \frac{n(n-1)(n-2r+1)(n-2r)(n-2r-1)\ldots3\cdot2\cdot1}{(n-2r)(n-2r-1)\cdot\ldots\cdot3\cdot2\cdot1} = \frac{n!}{(n-2r)!}\ldots\ldots\ldots(4)$

And $\quad 2\cdot4\cdot6\ldots2r = (2\cdot1)(2\cdot2)(2\cdot3)\ldots(2\cdot r) = 2^r\cdot r!\ldots\ldots\ldots(5)$

Finally, $(2n-1)(2n-3)\ldots(2n-r+1) = \frac{(2n)(2n-1)(2n-3)\ldots(2n-2r+2)(2n-2r+1)}{(2n)(2n-2)(2n-4)\ldots(2n-2r+2)}\times\frac{(2n-2r)!}{(2n-2r)!}$

$$= \frac{(2n)(2n-1)(2n-3)\ldots(2n-2r+2)(2n-2r+1)(2n-2r)(2n-2r-1)\ldots3\cdot2\cdot1}{2\cdot n\cdot2(n-1)(n-2)\ldots2(n-r+1)(2n-2r)!}$$

$$= \frac{(2n)!}{2^n\cdot n(n-1)(n-2)\ldots(n-r+1)(2n-2r)!} = \frac{(2n)!}{2^n(2n-2r)!}\times\frac{(n-r)(n-r-1)\ldots3\cdot2\cdot1}{n(n-1)(n-2)\ldots(n-r)(n-r-1)\ldots3\cdot2\cdot1}$$

$$= \frac{(2n)!}{2^n(2n-2r)!}\times\frac{(n-r)!}{n!}\ldots\ldots\ldots(6)$$

Using (3), (4), (5) and (6), the general term (2) becomes

$$\frac{(2n)!}{2^n\cdot n!}(-1)^r\cdot\frac{n!}{(n-2r)!}\times\frac{1}{2^r r!}\times\frac{2^n(2n-2r)!\,n!}{(2n)!\,(n-r)!}x^{n-2r}$$

i.e $\quad = (-1)^r\frac{(2n-2r)!}{2^n r!(n-r)!(n-2r)!}x^{n-2r}\ldots\ldots\ldots(7)$

Since (1) is polynomial of degree n, r must be chosen so that $n-2r \geq 0$, i.e., $r \leq \frac{n}{2}\cdot$

Thus, if n is even, $r$goes from 0 to $\frac{1}{2}n$ while if n is odd r goes from 0 to $\frac{1}{2}(n-1)$; that is, for the complete polynomial (1), r goes from 0 to $[\frac{1}{2}n]$, where

$$\left[\frac{1}{2}n\right] = \begin{cases}\frac{n}{2}, & if\ \ n = 2k\\ \frac{n-1}{2}, & if\ \ n = 2k-1\end{cases}$$

Hence the Legendre's polynomial of degree n is given by

$$p_n(x) = \sum_{r=0}^{\left[\frac{n}{2}\right]}(-1)^r\frac{(2n-2r)!}{2^r r!\,(n-r)!\,(n-2r)!}x^{n-2r}\ldots\ldots\ldots(8)\quad[5].$$

3. Kinds of Legendre's function. here is two kind of Legendre's function as follows:

3.1 Legendre's function of the first kind or Legendre's polynomial of degree n.

the solution of Legendre's equation is called Legendre's function When n is positive integer and $a = \frac{1\cdot3\cdot5(2n-1)}{n!}$, the solution (11) is denoted by $p_n(x)$ and is called Legendre's function of the first kind.

$\therefore \quad p_n(x) = \frac{1\cdot3\cdot5(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot4\cdot(2n-1)(2n-3)}x^{n-4} - \cdots\right]\ \ldots\ldots\ldots(1)$

$p_n(x)$ a terminating series and gives what are called Legendre's polynomials for different values of n. We can write $p_n(x) = \sum_{r=0}^{\left[\frac{n}{2}\right]}(-1)^r\frac{(2n-2r)!}{2^r r!(n-r)!(n-2r)!}x^{n-2r}$ where,

$$\left[\frac{1}{2}n\right] = \begin{cases}\frac{n}{2}, & if\ \ n = 2k\\ \frac{n-1}{2}, & if\ \ n = 2k-1\end{cases}\quad[3].$$

3.2 **Legendre's function of the second kind.** This is denoted and define by

$y = \frac{n!}{1\cdot3\cdot5\cdot\ldots\cdot(2n+1)}\left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-(n+3)} + \frac{n(n+1)(n+2)(n+3)(n+4)}{2\cdot4\cdot(2n+3)(2n+5)}x^{-(n+5)} + \cdots\right].\ldots\ldots\ldots(2)$

Example 3.2.1 if $n = 0, 1, 2, 3, 4$ *and* 5 in result (1), then find $p_0(x)$, $p_1(x)$, $p_2(x)$, $p_3(x)$, $p_4(x)$ and $p_5(x)$

Soluution.

$p_0(x) = \frac{1}{0!}x^0 = 1$,

$p_1(x) = \frac{1}{1!}x^1 = x,$
$p_2(x) = \frac{1\cdot 3}{2!}\left[x^2 - \frac{2\cdot 1}{2\cdot 3}x^0\right] = \frac{1}{2}(3x^2 - 1),$
$p_3(x) = \frac{1\cdot 3\cdot 5}{3!}\left[x^3 - \frac{3\cdot 2}{2\cdot 5}x^1\right] = \frac{1}{5}(5x^3 - 3x),$
$p_4(x) = \frac{1\cdot 3\cdot 5\cdot 7}{4!}\left[x^4 - \frac{4\cdot 3}{2\cdot 7}x^2 + \frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 4\cdot 7\cdot 5}x^0\right] = \frac{1}{8}(35x^4 - 30x^2 + 3)$ and
$p_5(x) = \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{5!}\left[x^5 - \frac{5\cdot 4}{2\cdot 9}x^3 + \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{2\cdot 4\cdot 9\cdot 7}x^1\right] = \frac{1}{8}(63x^5 - 70x^3 + 15x).$

Example 3.2.2 Express $2 - 3x + 4x^2$ in terms of Legendre's polynomials.
Solution. We have $1 = p_0(x),\ p_1(x) = x,\ p_2(x) = \frac{1}{2}(3x^2 - 1) \Rightarrow x^2 = \frac{[2p_2(x)+1]}{3}$ ……(1)
Now, $2 - 3x + 4x^2 = 2p_0(x) - 3p_1(x) + \frac{4}{3}\times[2p_2(x) + 1]$, by (1)

$$= 2p_0(x) - 3p_1(x) + \frac{8}{3}\times p_2(x) + \frac{4}{3}\times p_0(x)$$
$$= \frac{10}{3}\times p_0(x) - 3p_1(x) + \frac{8}{3}\times p_2(x)$$

Example 3.2.3 Express $x^4 + 2x^3 + 2x^2 - x - 3$ in terms of Legendre's polynomials.
Solutio. We have
$p_0(x) = 1,$
$p_1(x) = x,$
$p_2(x) = \frac{1}{2}(3x^2 - 1),$
$p_3(x) = \frac{1}{5}(5x^3 - 3x)$ and
$p_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3).$
$\Rightarrow x^4 = \frac{8}{35}\times p_4(x) + \frac{6}{7}\times x^2 - \frac{3}{35}$, ……… (1)
$x^3 = \frac{2}{5}p_3(x) + \frac{3}{5}x$, ……… (2)
$x^2 = \frac{2}{3}\times p_2(x) + \frac{1}{3}$ ……… (3)
$x = p_1(x)$ and $1 = p_0(x)$ ……… (4)
Now, $x^4 + 2x^3 + 2x^2 - x - 3 = x^4 + 2x^3 + 2x^2 - x - 3 + 2\left[\frac{2}{5}p_3(x) + \frac{3}{5}x\right] + 2x^2 - x - 3$

$$= \frac{8}{35}p_4(x) + \frac{4}{5}p_3(x) + \frac{20}{7}x^2 + \frac{1}{5}x - \frac{108}{35}$$
$$= \frac{8}{35}p_4(x) + \frac{4}{5}p_3(x) + \frac{20}{7}\left[\frac{2}{3}\times p_2(x) + \frac{1}{3}\right] + \frac{1}{5}p_1(x) - \frac{108}{35}$$

Using (2), (3) *and* (4) we get

$$= \frac{8}{35}p_4(x) + \frac{4}{5}p_3(x) + \frac{40}{21}\times p_2(x) + \frac{1}{5}p_1(x) - \frac{224}{105}p_0(x)$$

4. Generating for Legendre polynomials.

Theorem 4.1. Tshow that $(1 - 2xz + z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} z^n p_n(x), |x| \le 1, \quad |z| \le 1$ or to show that $p_n(x)$ is the coefficient of z^n in the exapansion of $(1 - 2xz + z^2)^{-\frac{1}{2}}$ in assending powers of z.
Note: $(1 - 2xz + z^2)^{-\frac{1}{2}}$ is called generating function for Legendre polynomial $p_n(x)$
Proof: Since $|z| \le 1$ and $|x| \le 1$, we have

$$(1 - 2xz + z^2)^{-\frac{1}{2}} = [1 - z(2x - z)]^{-\frac{1}{2}}$$
$$= 1 + \frac{1}{2}z(2x - z) + \frac{1\cdot 3}{2\cdot 4}z^2(2x - z)^2 + \cdots + \frac{1\cdot 3\cdot\ldots\cdot(2n-3)}{2\cdot 4\cdot\ldots\cdot(2n-2)}z^{n-1}(2x - z)^{n-1} + \frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}z^n(2x - z)^n \ldots\ldots (1)$$

Now, the coefficient of z^nin $\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}z^n(2x - z)^n = \frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}(2x)^n = \frac{1\cdot 3\cdot\ldots\cdot(2n-1)\cdot 2^n\cdot x^n}{(2\cdot 1)(2\cdot 2)(2\cdot 3)\cdot\ldots\cdot(2\cdot n)} = \frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2^n\cdot n!}2^n\cdot x^n$

$$= \frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{n!}x^n \ldots\ldots\ldots (2)$$

Again, the coefficient of z^n in $\frac{1\cdot 3\cdot\ldots\cdot(2n-3)}{2\cdot 4\cdot\ldots\cdot(2n-2)}z^{n-1}(2x - z)^{n-1} = \frac{1\cdot 3\cdot\ldots\cdot(2n-3)}{(2\cdot 1)(2\cdot 2)\cdot\ldots\cdot 2(n-1)}\{-(n-1)(2x)^{n-2}\}$

$= \frac{1\cdot3\cdot\ldots\cdot(2n-3)}{2^n\cdot1\cdot2\cdot3\cdot\ldots\cdot(n-1)} \frac{(2n-1)}{n} \frac{n}{(2n-1)} [(n-1)2^{n-2} \times x^n - 2]$, on multiplying and dividing by $\frac{2n-1}{n}$

$$\frac{1\cdot3\cdot\ldots\cdot(2n-1)}{n!}\frac{n(n-1)}{2(2n-1)}x^{n-2} \ldots\ldots\ldots (3)$$

And so on. Using (2) and (3) .... We see that the coefficient of z^n in the expansion of $(1-2xz+z^2)^{-\frac{1}{2}}$ namely (1) is given by $\frac{1\cdot3\cdot5(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot4\cdot(2n-1)(2n-3)}x^{n-4} - \cdots\right]$ i.e., $p_n(x)$, by definition of Legendre polynomial.

We find that $p_1(x)$, $p_2(x), \ldots$ will be the coefficient of $z, z^2, \ldots$ in the expansion of $(1-2xz+z^2)^{-\frac{1}{2}}$. Thuse, we may write $(1-2xz+z^2)^{-\frac{1}{2}} = 1 + zp_1(x) + z^2p_2(x) + \cdots + z^np_n(x) + \cdots$ or $(1-2xz+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} z^n p_n(x)$.

Example 4.1 prove that:

$$1 + \frac{1}{2}p_1(cos\theta) + \frac{1}{3}p_2(cos\theta) + \cdots = log\left[\frac{(1+sin\frac{1}{2}\theta)}{sin\frac{1}{2}\theta}\right]$$

Solution. From the generating function, $(1-2xz+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} z^n p_n(x) \ldots\ldots\ldots (1)$

Integrating (1) w.r.t z from 0 to 1,

$$\sum_{n=0}^{\infty}\int_0^1 z^n p_n(x)dz = \int_0^1 \frac{dz}{\sqrt{1-2xz+z^2}} \ldots\ldots\ldots (2)$$

Replacing x by $cos\theta$ on both sides, (2) gives

$$\sum_{n=0}^{\infty} p_n(cos\theta)\int_0^1 z^n dz = \int_0^1 \frac{dz}{\sqrt{1-2xz+z^2}}$$

$$\sum_{n=0}^{\infty} p_n(cos\theta)\left[\frac{z^{n+1}}{n+1}\right]_0^1 = \int_0^1 \frac{dz}{\sqrt{[(z-cos\theta)^2+sin^2\theta]}}$$

or $\sum_{n=0}^{\infty} \frac{p_n(cos\theta}{n+1} = \left[\log\left\{(z-cos\theta) + \sqrt{[(z-cos\theta)^2 + sin^2\theta]}\right\}\right]_0^1$

$= \log\left\{(1-cos\theta) + \sqrt{[(1-cos\theta)^2 + sin^2\theta]}\right\} - log(1-cos\theta)$

$= \log\left\{(1-cos\theta) + \sqrt{2(1-cos\theta)}\right\} - \log(1-cos\theta)$

$$= \log\frac{(1-cos\theta) + \sqrt{2}\sqrt{1-\cos\theta}}{1-cos\theta} = \log\frac{\sqrt{(1-cos\theta)}\sqrt{(1-cos\theta)} + \sqrt{2}\sqrt{1-\cos\theta}}{\sqrt{(1-cos\theta)}\sqrt{(1-cos\theta)}}$$

$$= \log\frac{\sqrt{(1-cos\theta)} + \sqrt{2}}{\sqrt{(1-cos\theta)}} = \log\frac{\sqrt{(2\sin^2\frac{1}{2}\theta)} + \sqrt{2}}{\sqrt{(2\sin^2\frac{1}{2}\theta)}} = log\frac{(1+sin\frac{1}{2}\theta)}{sin\frac{1}{2}\theta}$$

$$\therefore \quad \frac{p_0(cos\theta)}{1} + \frac{1}{2}p_1(cos\theta) + \frac{1}{3}p_2(cos\theta) + \cdots = log\frac{(1+sin\frac{1}{2}\theta)}{sin\frac{1}{2}\theta}$$

Or

$$1 + \frac{1}{2}p_1(cos\theta) + \frac{1}{3}p_2(cos\theta) + \cdots = log\frac{(1+sin\frac{1}{2}\theta)}{sin\frac{1}{2}\theta}$$

Example 4.2 Prove that:

$(i)\cdot \quad p_1(1) = 1$

$(ii)\cdot p_n(-1) = (-1)^n$

$(iii)\cdot p_n'(1) = \frac{1}{2}n(n+1)$.

$(iv)\cdot p_n'(-1) = (-1)^{n-1} \times \frac{1}{2}n(n+1)$.

$(v)\cdot p_n(-x)=(-1)^n p_n(x)$. Deduce $p_n(-1)=(-)^n$.

Solution. The generating function formula is $(1-2xz+z^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty} z^n p_n(x), |x|\le 1, \quad |z|\le 1 \dots\dots\dots (1)$

part $(\boldsymbol{i})\cdot$ Putting $x=1$ in (1), we get

$(1-2z+z^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty} z^n p_n(1)$, or $(1-z)^{-1}=\sum_{n=0}^{\infty} z^n p_n(1)$.

Since $|z|\le 1$, the binomial theorem can be used for expansion of $(1-z)^{-1}$.

$$\therefore \qquad 1-z+z^2-\cdots+(-1)^n z^n+\cdots=\sum_{n=0}^{\infty} z^n p_n(1). \dots\dots\dots (2)$$

Equating the coefficient of z^n from both sides, (2) gives $p_n(1)=1$

Part $(\boldsymbol{ii})\cdot$ Putting $x=-1$ in (1), we have as before

$(1+2z+z^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty} z^n p_n(-1)$, or $(1-z)^{-1}=\sum_{n=0}^{\infty} z^n p_n(-1)$.

Or $\qquad 1-z+z^2-\cdots+(-1)^n z^n+\cdots=\sum_{n=0}^{\infty} z^n p_n(-1). \dots\dots\dots (3)$

Equating the coefficient of z^n from both sides, (3) gives $p_n(-1)=(-1)^n$.

Part$(\boldsymbol{iii})\cdot$ Since $p_n(x)$ satisfies Legendre's equation $(1-x^2)y''-2xy'+n(n+1)y=0$, we get
$(1-x^2)p_n''(x)-2xp_n'(x)+n(n+1)p_n(x)=0 \dots\dots\dots (4)$
Putting $x=1$ in (4) and using $p_n(1)=1$, we get
$0-2p_n'(1)+n(n+1)=0$ or $p_n'(1)=\frac{1}{2}n(n+1)$.

Part $(\boldsymbol{iv})$. Putting $x=-1$ in (4) and using $p_n(-1)=(-1)^n$, we get
$0+2\mathrm{p}_n'(-1)+n(n+1)(-1)^n=0$ or $p_n'(-1)=-(-1)^n\times\frac{1}{2}n(n+1)$.
Or $p_n'(-1)=(-1)=(-1)^{n-1}\times\frac{1}{2}n(n+1) \qquad [\because\ -(-1)^n=-(-1)^{n-1}(-1)=(-1)^{n-1}$

Part $(\boldsymbol{v})$. Replace x by $-x$ in (1), $(1+2xz+z^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty} z^n p_n(-x) \quad \dots\dots\dots (5)$
Next, replacing z by $-z$ in (1), $(1+2xz+z^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}(-z)^n p_n(x) \quad \dots\dots\dots (6)$
From (5) and (6), $\qquad \sum_{n=0}^{\infty} z^n p_n(-x)=\sum_{n=0}^{\infty}(-z)^n p_n(x) \quad \dots\dots\dots (7)$
Equating the coefficients of z^n from both sides of (7), we get

$$p_n(-x)=(-1)^n p_n(x) \dots\dots\dots (8)$$

Deduction. Replace x by 1 and noting that $p_n(1)=1$, (8) gives $p_n(-1)=(-1)^n$

Note. When n is odd, $(-1)^n=-1$ and so (8) becomes $p_n(-x)=-p_n(x)$. Thus, $p_n(x)$ is an odd function of x when n is odd. Similarly, $p_n(x)$ is an even function of x when n is even [1].

Orhogonal property 4.1. If p_m and p_n are legendre polynomials,

$$\int_{-1}^{1} p_m(x)p_n(x)dx=0 \text{ if } m\neq n \qquad \text{and} \qquad \int_{-1}^{1} p_n^2(x)dx=\frac{2}{2n+1} \ \ if \ \ m=n$$

Proof **1.** Equation $(1-x^2)y''-2xy'+n(n+1)y=0$ can be written as

$$\frac{d}{dx}[(1-x^2)p_n']=-n(n+1)p_n$$
$$\frac{d}{dx}[(1-x^2)p_m']=-m(m+1)p_{m.}$$

Multiply the first relation by p_m and the second relation by p_n and subtract the resulting expressions. Hence, we get

$$\frac{d}{dx}[(1-x^2)(p_m p_n'-p_n p_m')]=[m(m+1)-n(n+1)]p_n p_m.$$

Now, integrate between the limits -1 and 1. The conclusion follows. ▯

The equation $\int_{-1}^{1} p_m(x)p_n(x)dx = 0$ if $m \neq n$ stisfied by $p_n,\ p_m,\ m \neq n$ is called orthogonal relation for p_n with weight 1. Therefore legendre polynomial form an orthogonal set of function with weight function unity on $[-1,1]$. The orthogonal property of p_n is crucially used in the expansion of given function g defined and continuous on $[-1,1]$ in terms of p_n.

***Proof* 2.** Let us denote $V = (x^2 - 1)^n$ then,

$$\int_{-1}^{1} p_n^2(x)dx = \int_{-1}^{1} \left(\frac{1}{n!\,2^n}\right)^2 \frac{d^n}{dx^n} V(x) \left(\frac{1}{n!\,2^n}\right)^2 \frac{d^n}{dx^n} V(x)$$

Let us evaluate the integral given below.

$$I = \int_{-1}^{1} \frac{d^n}{dx^n} V(x) \frac{d^n}{dx^n} V(x).$$

Note thate

$$V^{(m)}(-1) = V^{(m)}(1) = 0, if \quad 0 \leq m < n.$$

We successively integrate by parts the integral I and get

$$I = \int_{-1}^{1} \left[\frac{d^{2n}}{dx^{2n}} V(x)\right] (-1)^n V(x)dx = (2n)! \int_{-1}^{1} (1 - x^2)^n dx.$$

With the help of the transformation $t = cos\theta$ and using the formula for $\int_0^{\frac{\pi}{2}} \sin^m \theta d\theta,$ we arrive at

$$\int_{-1}^{1} p_n^2(x)dx = \frac{2}{2n+1} \quad [5].$$

Conclusion

In this article we explained the concept of notion of legendre's equation, first and second kind of legendre polynomial, generating function and orthogonal property which are important in applied mathemtics field.

Reference

[1]. M.D. RaisingHania, (2019), "Ordinary and Partial Differentail Equations", S.chand and Company Limited,Ram,Nagar, New Deli, (9.1-9.10).

[2]. P. Agarwal, Ravi and Donal O'Regan, (2009), "Ordinary and Partial Differential Equation with Special Function , Forier Series, and Boundary Value Problems", springer, New York, 74.

[3]. P.K. Mittal, (2008), "Mathematics For Degree Student B.Sc. second year", S.chand and Company Pvt.LTD,RamNagar, New Deli, (314).

[4]. Said-Houari, Belkacem, (2015), "Differential Equation Methods and Aplication", United Arab Emirates 135.

[5]. SGDeo, V Raghavendra, Rasmita Kar, V Lakshmi Kantham (2015), "atextbook of Ordinary Differential Equation", McGraw Hill Education(india) private Limeted, New Delhi, (239 − 243).