MATHEMATICAL MODELLING OF A QUADROTOR

ANDRIAMANALINA Ando Nirina[1], RANDRIAMITANTSOA Andry Auguste[2], RANDRIAMITANTSOA Paul Auguste[3]

[1] *Doctoral School in Science and Technology of Engineering and Innovation (ED – STII)*
[1] *Research Laboratory in Telecommunication, Automation, Signal and Images*
[1] *University of Antananarivo, BP 1500, Antananarivo 101 - Madagascar.*
[1] *Doctoral School in Science and Technology of Engineering and Innovation (ED – STII)*
[1] *Research Laboratory in Telecommunication, Automation, Signal and Images*
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[1] *University of Antananarivo, BP 1500, Antananarivo 101 - Madagascar.*

ABSTRACT

For quadrotor, the motion is controlled by varying each of four motor speeds to obtain a desired effect that causes movement. The possible maneuvers of the drone are multiple such as movements of translation and rotation and levitation. Translational motion always depends on rotational motion. In this research paper, modelisation of a quadrotor with motion equation, state space starting from basic Newtonian equations and simulation with Matlab are realized.

Keyword : *UAV, roll, pitch, yaw, motion*

1. INTRODUCTION

Unmanned aerial vehicles (UAVs), usually termed as drones are a robotic aerial vehicle capable of carrying out a mission autonomously. Quadrotor drone is a type of helicopter with four rotors. Technology allowed increasingly the development of small drones, with moderate cost and easy to use. To prevent the device from rotating on itself, two rotors spinning clockwise and two counter clockwise. Here, modelisation of UAV is motion equation and space states with six degrees of freedom.

2. POSITION AND ORIENTATION OF THE QUADCOPTER

Knowledge of position and orientation of quadrotor drone, require two reference system. The first (O'X'Y'Z') is mobile coordinate with the drone, relate his orientation. The second (OXYZ) is a fixed coordinate called inertial is independent of the position of the drone. For the first reference, the origin O' of coordinate coincides with the center of gravity of the quadrotor [1].

To illustrate respectively yaw motion, pitch motion and roll motion, three angles namely ψ, θ and φ are used.

These angles are expressed by:

$$-\pi < \psi < \pi \qquad (01)$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2} \qquad (02)$$

$$-\pi/2 < \varphi < \pi/2 \qquad (03)$$

To know the orientation of the mobile reference with respect of the fixed reference according to values of ψ, θ and φ angles, a rotation matrix $R_{\psi\theta\varphi}$ is use.

$$R_{\psi\theta\varphi} = \begin{bmatrix} c(\varphi)c(\psi) + s(\varphi)s(\theta)s(\psi) & s(\varphi)s(\theta)c(\psi) - c(\varphi)\,s(\psi) & s(\varphi)c(\theta) \\ c(\theta)s(\psi) & c(\theta)c(\psi) & -s(\theta) \\ c(\varphi)s(\theta)s(\psi) - s(\varphi)\,c(\psi) & s(\varphi)\,s(\psi) + c(\varphi)s(\theta)c(\psi) & c(\varphi)c(\theta) \end{bmatrix} \qquad (04)$$

Where $c = cos$ and $s = sin$

Let be considered a vector $\vec{H} = (x \quad y \quad z)^T$ defined in the fixed reference where the mobile reference initially coincides. After three rotations of $\vec{H}$ along the ψ, θ and φ, this vector is represented by another vector $\overrightarrow{H_{\psi\theta\varphi}}$ and expressed in the fixed reference by:

$$\overrightarrow{H_{\psi\theta\varphi}} = x'\vec{i} + y'\,\vec{j} + z'\,\vec{k} \qquad (05)$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R_{\psi\theta\varphi}\vec{H} = R_{\psi\theta\varphi}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad (06)$$

If, origin of two reference coincide, the mobile reference the same angle rotations as $\vec{H}$ then in the mobile reference, $\overrightarrow{H_{mobil}}$ is expressed by:

$$\overrightarrow{H_{mobile}} = x\vec{\imath'} + y\,\vec{j'} + z\,\vec{k'} \qquad (07)$$

3. DRONE MOTION

With high manoeuvrability, drone is related through five movements as, vertical, yaw, roll, pitch and translation motion [2].

3.1 Vertical motion

The vertical motion corresponds to the ascent or descent of the quadcopter. It is achieved by increasing or decreasing simultaneously the speed of all four motors.

3.2 Yaw motion

The yaw motion is a movement around vertical axis. It is obtained by increasing the speed of the normal pitch propellers and proportionally decreasing the speed of the opposite pitch propellers. All the four rotors contribute in yawing moment.

3.3 Rolling motion

Roll motion is movement around X’ axis. This movement is achieved by increasing the speed of one propeller over Y’ axis and decreasing the speed of the opposite propeller.

3.4 Pitch Movement

This motion is movement around Y’ axis. It is very similar to roll and is achieved by increasing the speed of one propeller over X’ axis and decreasing the speed of the opposite propeller.

3.5 Translational motion

A translational movement following right or left is obtained by tilting the drone by a rolling motion and by increasing all the thrust produced compared to a stationary flight to keep the importance of the following component Z. As well as left or right, forward or backward translational motion is obtained by tilting the drone by a pitching movement and increasing all the thrust.

4. EQUATION OF MOTION QUADROTOR

4.1 Forces

Based on the Newton-Euler model, during the movement of the drone, several forces and moments interact.

- The weight of the quadrotor $\vec{P} = m\vec{g}$
- The thrust forces represent the coefficient of lift and the rotational velocity of the i[th] engine.

$$F_i = b\omega_i^2$$

Where, b is the thrust factor.

In the mobile reference, the total thrust is expressed by:

$$\vec{F} = \sum_{i=1}^{4} \vec{F_i} = \sum_{i=1}^{4} F_i \vec{k'} \quad (08)$$

- The drag forces that oppose the engine torque following the propellers are expressed by [3]:

$$T_i = d\omega_i^2 \quad (09)$$

Where, d is the drag factor which depends on the manufacture of the propeller.

Independently of the gyroscopic effects, the moments due to the rotations around OX', OY', OZ' and generated by the drag torques are expressed by:

$$\mathcal{M}_x = lb(\omega_4^2 - \omega_2^2) \quad (10)$$

$$\mathcal{M}_y = lb(\omega_3^2 - \omega_1^2) \quad (11)$$

$$\mathcal{M}_z = d(\omega_1^2 - \omega_2^2 + \omega_3^2 - \omega_4^2) \quad (12)$$

Where l is the distance between any rotor and the center of the drone.

4.3 Translation motion equation

The translation motion equations along the OX, OY and OZ axes are expressed by:

$$\ddot{x} = \frac{[\sin(\psi)\sin(\varphi) + \cos(\psi)\sin(\theta)\cos(\varphi)]F}{m} \quad (13)$$

$$\ddot{y} = \frac{[\sin(\psi)\sin(\theta)\cos(\varphi) - \cos(\psi)\sin(\varphi)]F}{m} \quad (14)$$

$$\ddot{z} = -g + \frac{\cos(\varphi)\cos(\theta)F}{m} \quad (15)$$

Where g and represent, the gravity and the mass of the drone.

For rotational motion equation, we have [4]:

$$J\begin{pmatrix}\ddot{\psi}\\ \ddot{\theta}\\ \ddot{\varphi}\end{pmatrix} = \sum \mathcal{M}_{ext} \quad (16)$$

Where J is the inertia matrix defined by:

$$J = \begin{pmatrix} I_X & 0 & 0\\ 0 & I_Y & 0\\ 0 & 0 & I_Z\end{pmatrix} \quad (17)$$

Where I_X, I_Y and I_Z represent respectively moment of OX, OY and OZ.

The acceleration according to the angles of rotation is expressed by:

$$\begin{pmatrix}\ddot{\psi}\\ \ddot{\theta}\\ \ddot{\varphi}\end{pmatrix} = \begin{bmatrix} \frac{lb(\omega_4^2 - \omega_2^2)}{I_X}\\ \frac{lb(\omega_3^2 - \omega_1^2)}{I_Y}\\ \frac{d(\omega_1^2 - \omega_2^2 + \omega_3^2 - \omega_4^2)}{I_Z}\end{bmatrix} \quad (18)$$

5. QUADCOPTER STATE SPACE REPRESENTATION

The vector that contains the quadcopter state variables is:

$$X_d = [x \quad \dot{x} \quad y \quad \dot{y} \quad z \quad \dot{z} \quad \psi \quad \dot{\psi} \quad \theta \quad \dot{\theta} \quad \varphi \quad \dot{\varphi}]^T \quad (19)$$

The vector that contains the input variables or commands is defined by:

$$U = [U_1 \quad U_2 \quad U_3 \quad U_4 \quad U_5]^T \qquad (20)$$

Where:

$$U_1 = F_1 \qquad (21)$$
$$U_2 = F_2 \qquad (22)$$
$$U_3 = F_3 \qquad (23)$$
$$U_4 = F_4 \qquad (24)$$
$$U_5 = 1 \qquad (25)$$

The vector that contains the system output variables are:

$$Y_d = [x \quad y \quad z \quad \psi \quad \theta \quad \varphi]^T \qquad (26)$$

The state equation that connects the input, state and output vectors are:

$$\dot{x}_d(t) = A\, x_d(t) + B\, u(t) \qquad (27)$$
$$\dot{y}_d(t) = C\, x_d(t) + D\, u(t) \qquad (28)$$

$$\begin{pmatrix} \dot{x} \\ \ddot{x} \\ \dot{y} \\ \ddot{y} \\ \dot{z} \\ \ddot{z} \\ \dot{\psi} \\ \ddot{\psi} \\ \dot{\theta} \\ \ddot{\theta} \\ \dot{\varphi} \\ \ddot{\varphi} \end{pmatrix} = \begin{pmatrix} 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \\ y \\ \dot{y} \\ z \\ \dot{z} \\ \psi \\ \dot{\psi} \\ \theta \\ \dot{\theta} \\ \varphi \\ \dot{\varphi} \end{pmatrix} + \begin{pmatrix} 0&0&0&0&0 \\ \frac{a_1}{m}&\frac{a_1}{m}&\frac{a_1}{m}&\frac{a_1}{m}&0 \\ 0&0&0&0&0 \\ \frac{a_2}{m}&\frac{a_2}{m}&\frac{a_2}{m}&\frac{a_2}{m}&0 \\ 0&0&0&0&0 \\ \frac{a_3}{bm}&\frac{a_3}{bm}&\frac{a_3}{bm}&\frac{a_3}{bm}&-g \\ 0&0&0&0&0 \\ 0&-\frac{l}{I_X}&0&\frac{l}{I_X}&0 \\ 0&0&0&0&0 \\ -\frac{l}{I_Y}&0&\frac{l}{I_Y}&0&0 \\ 0&0&0&0&0 \\ \frac{d}{bI_Z}&-\frac{d}{bI_Z}&\frac{d}{bI_Z}&-\frac{d}{bI_Z}&0 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} \qquad (29)$$

With

$$a_1 = \sin(\psi)\sin(\varphi) + \cos(\psi)\sin(\theta)\cos(\varphi) \quad (30)$$
$$a_2 = \sin(\psi)\sin(\theta)\cos(\varphi) - \cos(\psi)\sin(\varphi) \quad (31)$$
$$a_3 = \cos(\varphi)\cos(\theta) \qquad (32)$$

$$\begin{pmatrix} x \\ y \\ z \\ \psi \\ \theta \\ \varphi \end{pmatrix} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \\ y \\ \dot{y} \\ z \\ \dot{z} \\ \psi \\ \dot{\psi} \\ \theta \\ \dot{\theta} \\ \varphi \\ \dot{\varphi} \end{pmatrix} + \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} \qquad (33)$$

Then,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (34)$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{a_1}{m} & \frac{a_1}{m} & \frac{a_1}{m} & \frac{a_1}{m} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{a_2}{m} & \frac{a_2}{m} & \frac{a_2}{m} & \frac{a_2}{m} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{a_3}{bm} & \frac{a_3}{bm} & \frac{a_3}{bm} & \frac{a_3}{bm} & -g \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{l}{I_X} & 0 & \frac{l}{I_X} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{l}{I_Y} & 0 & \frac{l}{I_Y} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{d}{bI_Z} & -\frac{d}{bI_Z} & \frac{d}{bI_Z} & -\frac{d}{bI_Z} & 0 \end{pmatrix} \quad (35)$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad (36)$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (37)$$

Transfert function of quadrotor is defined by:

$$F(p) = C[pI - A]^{-1}B \quad (38)$$

6. SIMULATION

During simulation with MATLAB, all proprieties of quadrotor are defined by:

- $I_X = 0{,}025$
- $I_y = 0{,}025$
- $I_z = 0{,}03$
- $l = 0{,}25$;
- $b = 0{,}000001{,}9$
- $d = 0{,}000001{,}84$
- $m = 0{,}25$
- $g = 9{,}81$

- $a_1 = a_2 = a_3 = 1$

6.1 Transfert function

With state space (34), (35), (36) and (37) we have :

$$F_1(p) = \frac{4}{p^2}$$
$$F_2(p) = \frac{4}{p^2}$$
$$F_3(p) = \frac{4}{p^2}$$
$$F_4(p) = 0$$
$$F_5(p) = -\frac{10}{p^2}$$
$$F_6(p) = 0$$

6.2 Step response

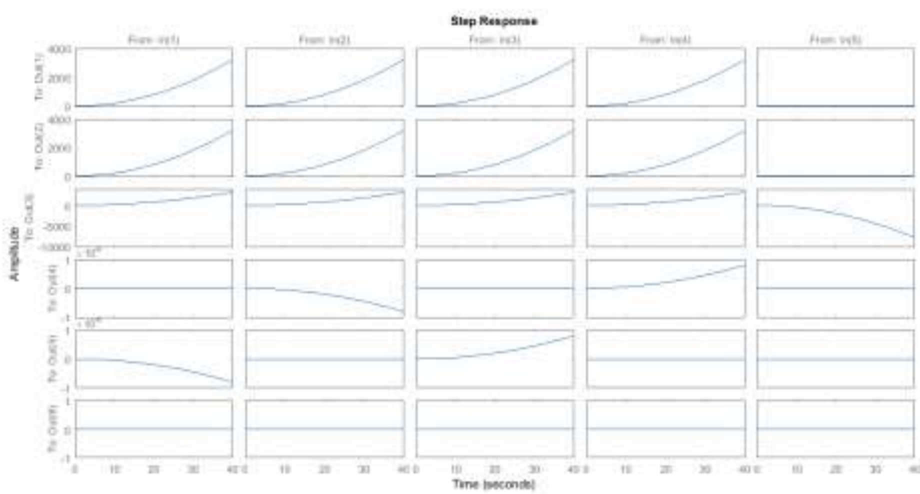

Fig -1: Step response

Here,

- Inputs are $U = [U_1 \quad U_2 \quad U_3 \quad U_4 \quad U_5]^T$
- Outputs variables are: $Y_d = [x \quad y \quad z \quad \psi \quad \theta \quad \varphi]^T$

7. RESULTS

Rotational and translational movements of a quadrotor depend on the rotor velocity, which generate the total thrust. The rotational movements are independent of the translational movements while the latter depends on the rotational movements. An increase in thrust is necessary during a translation because rolling or pitching decreases the altitude of the drone.

6. CONCLUSION

As with most robotic systems, the dynamic system of a drone is complex. According to the Newton-Euler model, precision of the rotational movements of the drone guarantees control of all the maneuvers performed. All, motion is controlled by varying each of four motor speeds. Drone is related through five movements as, vertical, yaw, roll, pitch and translation motion.

6. REFERENCES

[1]. A. Joukhadar, M. Alchehabi, A. Jejeh “Advanced UAVs non colinear control system an applications”, InTech, 2019.

[2]. F. Sabatino, “Quadrotor control: modeling, nonlinear control design, and simulation”, KTH Electrical Engineering, 2015.

[3]. Z. Tahir, W. Tahir, SA Liaqat, A.-J. Van Der Veen, “State space system modeling of a quadcopter UAV”.

[4]. S. Zouaoui, *“Contribution to the modeling and control of a miniature drone*», Thesis, Djillali University, 2018.