MATHEMATICAL MODELLING OF PIEZOELECTRIC HARVESTING FROM BRIDGE VIBRATION UNDER MOVING LOAD

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ABSTRACT

This paper shows a possibility of harvesting electrical energy from mechanical vibration. In this way, a piezoelectric material is used to harvest energy from bridge vibration under a moving load. The response of bridge under moving load is studied first, and second the energy harvesting with a piezoelectric element, and third the combination of the two phenomena. The bridge is modelled as a simple beam with length l, supported at both the ends, and traversed by a force P moving with a constant velocity c. Its behaviour is described by the Euler-Bernoulli differential equation. Modal superposition is used to solve this equation, and a transformation is made from displacement coordinate to modal coordinate. A new differential equation is obtained and a resolution in the frequency domain by Fourier transform and Laplace transform is used to study the deflection of the bridge. A piezoelectric model in one dimension is also considered. The model is composed of a seismic mass that compresses all area of piezoelectric material. This means that the model is studied in longitudinal mode or mode33. The piezoelectric element is connected to a power harvesting circuit modelled as a single resistor R. A transformation in piezoelectric constitutive equation gives a differential equation of the electromechanical system. Laplace transform is used to solve the equation in frequency domain. The displacement and the generated voltage for the piezoelectric harvester are obtained. A mathematical study was done by combining the two models. The idea is to put a piezoelectric element directly under the bridge with glue. In this case, the deflection of the bridge is considered as a stress applied to the piezoelectric element. Numerical application is chosen for bridge and piezoelectric parameters. Results are obtained by varying the position of the harvester, the speed of the moving load and the length of the bridge. A maximum power between 99.9 µW and 438.20 µW is obtained with the bridge length between 25 m and 50 m.

Keyword: *Piezoelectric, Bridge vibration, Energy harvesting, Electrical energy, Fourier and Laplace transform*.

1. INTRODUCTION

Any small vibrations are present around us every day and they are just wasted if not exploited. The idea of this work is to look for a way to convert this vibratory energy into usable electrical energy. More sources of the vibratory energy are possible, such as mechanical energy caused by industrial machines, cars, and trains. There are also vibrations from large infrastructures such as bridge and building. Human movement like foot movement and the joints movement are also to be mentioned. A piezoelectricity is one phenomenon capable of converting the small vibrations to electrical energy. Research works were already made in this area. Modelling of the stress distribution in the power generation module under the traffic in roadway was investigated by Papagiannakis et al. [1]. The results show that the energy obtains can drive LED traffic lights and wireless sensors built into the pavement structure. Gao, Wang, Cao, Chen and Liu [2] have analysed a piezoelectric harvester placed on a rail to generate electrical energy from the acceleration of the rail. They use a cantilevered piezoelectric beam to visualize voltage and power. Jingjing Zhao and Zheng You [3] have made use of human motion to obtain electrical energy for a power portable sensor. The piezoelectric element is placed in the man's shoe. The energy harvesting under bridge vibration has also been the subject of several works. Jacopo Bonari and Paolo Valvo [4] have studied possibility of using of piezoelectric material to harvest the energy under a vibrations induced in bridges when vehicles go accross. They considered a cantilever beam of a piezoelectric element with the same first frequency of a bridge in numerical studies. Ye Zang [5] worked on the piezoelectric harvesting in infrastructures of a particular bridge structures. He use a harvester based on cantilever for the simulation. The results show the interests of using energy harvester in the field of civil engineering. In this work, a mathematical resolution of piezoelectric harvesting under bridge vibration is studied. The bridge model and piezoelectric harvesting model are studied separately first. The bridge is modelled as a simple beam with a vehicle moving with a constant speed crossing it. And then a piezoelectric harvester model is used to investigate the longitudinal deformation mode for one degree of freedom. And finally, the two models are combined to constitute the harvesting system.

2. MATHEMATICAL MODELLING OF THE BRIDGE RESPONSE UNDER CONSTANT SPEED MOVING LOAD

The bridge is modelled as a simple beam supported at both the end. The bridge has a length *l* and a load is moving across it with a constant speed *c*, from left to right. Only gravitational effect is considered because the weight of load is small compared to that of the beam. The beam has zero displacement at both ends. With this assumption, the governing differential equation of the model is described by of Bernoulli-Euler equation. The model is shown in Figure 1.

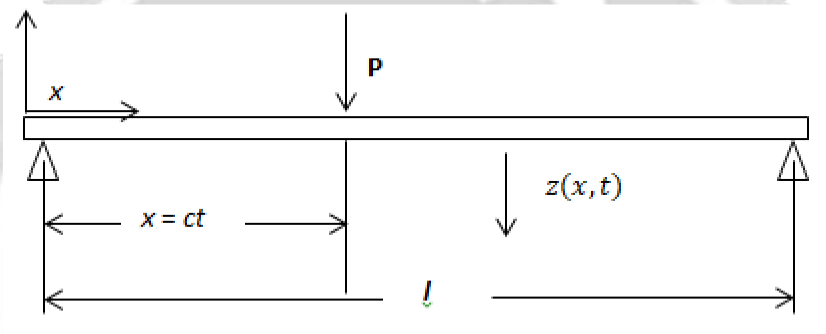

Fig -1: Bridge under a force P

At the time *t* and at the position *x*, the displacement or deflection $z(x,t)$ of the bridge is giving by equation:

$$EI\,\frac{\partial^4 z(x,t)}{\partial x^4} + \mu\,\frac{\partial^2 z(x,t)}{\partial t^2} + 2\mu\zeta_n\omega_n\frac{\partial\, z(x,t)}{\partial t} = p(x,t) \quad (1)$$

Where$E$, I, μ, ζ_n, ω_n are respectively the Young's modulus , the moment of inertia of the beam cross section, the mass per unit length, the damping coefficient of the beam and the natural frequency of the beam for n^{th} mode.

$p(x,t)$ can considered as :

$$p(x,t) = \begin{cases} \delta\,(x-ct)P, & 0 \le t \le t_d \;\; avec\; t_d = l/c \\ 0, & t_d > 0 \end{cases} \quad (2)$$

Using modal superposition, the solution of equation (1) can resolve with series form as:

$$z(x,t) = \sum_{n=1}^{\infty} Z_n\,(t)\,sin\left(\frac{n\pi x}{l}\right) \quad (3)$$

It is a transformation from displacement coordinates $z(x,t)$ to the modal coordinates $Z_n(t)$,
where $Z_n(t)$ is the undamped deflection mode shape *n* in the case of undamped free vibration.
The use of equation (3) in (1) and a mathematic manipulation [7] give differential equation in modal coordinates.

$$\ddot{Z}_n(t) + 2\zeta_n\omega_n\,\dot{Z}_n(t) + \omega_n^2\,Z_n(t) = \frac{2P}{\mu l}\sin(n\phi t) = f_n(t) \quad (4)$$

Where $\omega_n^2 = \frac{EJ}{\mu}\left(\frac{n\pi}{l}\right)^4$, $\phi = \frac{\pi c}{l}$ witches are respectively the natural frequency and the excitation frequency.

2.1 Frequency domain solution

Laplace transform of equation (4) with zero as the initial conditions give [7]:

$$(s^2 Z_n(p) - s\,Z_n(0) - \dot{Z}_n(0)) + 2\zeta_n\omega_n(s\,Z_n(p) - Z_n(0)) + \omega_n^2(Z_n(s)) = F_m(s)$$

$$Z_n(s) = \frac{1}{(s^2 + 2\zeta_n\omega_n s + \omega_n^2)} * F_n(s)$$

with $s = j\omega$

$$Z(\omega) = \frac{1}{(-\omega^2 + 2j\omega\zeta_n\omega_n + \omega_n^2)} * F_n(\omega)$$

The Fourier transforms $F_n(\omega)$ in the right hand side of equation (4) is:

$$F_n(\omega) = \frac{1}{\sqrt{2\pi}}\int f_n(t)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{2P}{l\mu}\,\frac{1}{2j}\left(\frac{e^{j(n\phi-\omega)T}}{j(n\phi-\omega)} + \frac{e^{j(n\phi-\omega)T}}{j(n\phi+\omega)} - \frac{1}{j(n\phi-\omega)} + \frac{1}{j(n\phi+\omega)}\right)$$

The solution of equation (1) in frequency domain is:

$$z(x,\omega) = \sum_{n=1}^{\infty}\frac{1}{(-\omega^2 + 2j\omega\zeta_n\omega_n + \omega_n^2)}\,\frac{1}{\sqrt{2\pi}}\,\frac{2P}{l\mu}\,\frac{1}{2j} \quad (5)$$

$$* \left(\frac{e^{j(n\phi-\omega)T}}{j(n\phi-\omega)} + \frac{e^{j(n\phi-\omega)T}}{j(n\phi+\omega)} - \frac{1}{j(n\phi-\omega)} + \frac{1}{j(n\phi+\omega)}\right) sin\left(\frac{n\pi x}{l}\right)$$

3. MATHEMATICAL MODELLING OF PIEZOELECRIC

A piezoelectric model in one dimension shown in Figure 2 is considered. The model is composed of a seismic mass that compresses all area of piezoelectric material. The deformation of piezoelectric element has a same direction of applied electric field. This means that the model is studied in longitudinal mode or mode33 [6]. The piezoelectric element is connected to a power harvesting circuit modelled as a single resistor R.

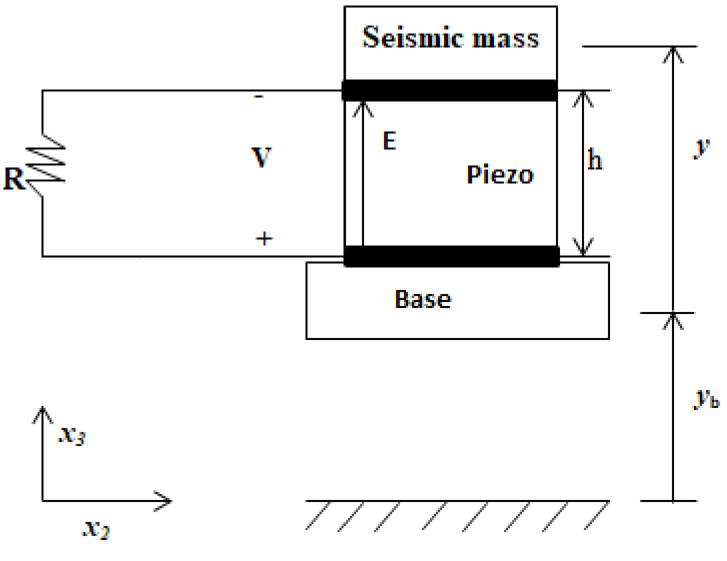

Fig -2: Electromechanical model of piezoelectric harvesting in 1D

For this model, the piezoelectric constitutive equation [6] in direct and inverse piezoelectric effects can be written as:

$$\begin{cases} T_3 = C_{33}^E . S_3 - e_{33}^t . E_{33} \\ D_3 = e_{33} . S_3 + \varepsilon_{33}^S . E_3 \end{cases} \quad (6)$$

D, *E*, *S*, *T*, *e*, ε, C^E, are respectively : the electric displacement which is produced, the applied electric field, the applied strain, the developed stress, the piezoelectric stress constant, the electric permittivity and the stiffness matrix. The superscript *E* and *S* are parameters at a constant electric field and strain.

Equation (6) can be transformed with a following definition:

- strain related to y and the thickness h : $S_3 = \frac{y}{h}$
- electric field related to voltage *v* and the thickness h : $E_3 = \frac{v}{h}$
- mass per cross section: $m_t = \frac{m}{A}$, where *m* is a total mass of the system
- stress which is the force per area : $T_3 = -\frac{m(\ddot{y}+\ddot{y}_b)}{A} = -m_t(\ddot{y} + \ddot{y}_b)$
- electric displacement of the charge per unit area : $D_3 = \frac{q}{A}$

Equation (6) can be rewritten as:

$$\begin{cases} m_t\ddot{y} + C_{33}^E \dfrac{y}{h} + e_{33}\dfrac{v}{h} = -m_t\ddot{y}_b \\ D_3 = e_{33}\dfrac{y}{h} - \varepsilon_{33}^S \dfrac{v}{h} = \dfrac{q}{A} \end{cases} \quad (7)$$

With other definitions of parameters:

- the *electromechanical* coupling $\theta = \frac{e_{33} A}{h}$
- the *stiffness* $k = \frac{C_{33}^E A}{h}$
- the *capacitance* $C_p = \frac{\varepsilon_{33}^S A}{h}$
- the developed voltage $v = i R = \frac{d_q}{dt} R$
- the viscous damping c_d proportional to velocity $\dot{y}$

Using the last parameters in (7), the final piezoelectric model equation is:

$$\begin{cases} m\ddot{y} + c_d\dot{y} + ky - \theta v = -m\ddot{y}_b \\ \theta\dot{y} + C_p\dot{v} + \dfrac{1}{R}v = 0 \end{cases} \quad (8)$$

3.1 Frequency domain solution of piezoelectric model

Laplace transform [7] of the first equation in (8) with the initial condition give:

$$m\ [s^2\, Y(s) - sy(0) - \dot{y}(0)] + c_d[sY(s) - y(0)] + kY(s) - \theta V(s) = -m\,[s^2\, Y_b(s) - sy_b(0) - \dot{y}_b(0)]$$

With

- $s = j\omega$
- *natural* frequency: $\omega_h = \sqrt{\frac{k}{m}}$
- the *damping*: $\zeta_h = \frac{c_d}{2m\omega_h}$,

The equation can be rewritten as:

$$(-\omega^2 + 2j\omega\zeta_h\omega_h + \omega_h^2)Y(\omega) - \frac{\theta}{m}V(\omega) = \omega^2\, Y_b(\omega)$$

Dividing by ω_h^2, and $\Phi_h = \frac{\omega}{\omega_h}$, the equation take the form :

$$[(1-\Phi_h^2)+2j\zeta_h\Phi_h]Y(\omega)-\frac{\theta}{k}V(\omega)=\Phi_h^2\,Y_b(\omega) \quad (9)$$

Laplace transform for the second equation with the initial condition give [7]:

$$\theta s[sY(s)-y(0)]+C_p[sV(s)-v(0)]+\frac{1}{R}V(s)=0$$

Let $s=j\omega$, and dividing by $C_p\omega_h$:

$$j\frac{\omega}{\omega_h}\frac{\theta}{C_p}Y(\omega)+\left(j\frac{\omega}{\omega_h}+\frac{1}{\omega_h C_p R}\right)V(\omega)=0$$

Let $\beta=\omega_h C_p R$

$$j\Phi_h\frac{\beta\theta}{C_p}Y(\omega)+(j\Phi_h\beta+1)V(\omega)=0 \quad (10)$$

Equations (9) and (10) can be written in the matrix form:

$$\begin{bmatrix}(1-\Phi_h^2)+2j\zeta_h\Phi_h & -\frac{\theta}{k}\\ j\Phi_h\frac{\beta\theta}{C_p} & (j\Phi_h\beta+1)\end{bmatrix}\begin{bmatrix}Y(\omega)\\ V(\omega)\end{bmatrix}=\begin{bmatrix}\Phi_h^2\,Y_b(\omega)\\ 0\end{bmatrix} \quad (11)$$

This equation gives respectively the displacement y and the voltage developed v with $Y(\omega)$ and $V(\omega)$ in frequency domain. In other words, this is the matrix form of piezoelectric harvesting in frequency domain for one dimension.

4. MATHEMATICAL MODELLING FOR THE TWO MODELS: PIEZOELECTRIC ELEMENT MOUNTED UNDER THE BRIDGE

The system studied here is the combination of two models previously discussed. The idea is to use a vibration of the bridge which is obtained with the moving load in it. This vibration is considered as the input stress to the piezoelectric energy harvester put under the bridge. The system is shown in Figure 3.

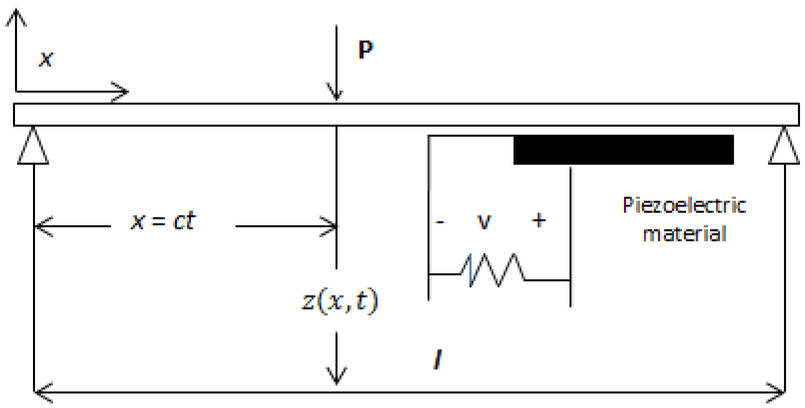

Fig - 3: Model « bridge piezoelectric harvesting »

4.1 Equation for the model

The displacement of the bridge $z(x,\omega)$ presented in equation (5) is nothing else than the base movement $Y_b(\omega)$ in the piezoelectric harvesting presented in equation (11).

$$z(x,\omega)\approx Y_b(\omega) \quad (12)$$

$Y_b(\omega)$ is obtained by matrix calculation using the equation (11). Let M be the matrix.

$$M=\begin{bmatrix}(1-\Phi_h^2)+2j\zeta_h\Phi_h & -\frac{\theta}{k}\\ j\Phi_h\frac{\beta\theta}{C_p} & (j\Phi_h\beta+1)\end{bmatrix}$$

$$\det M = (j\Phi_h)^3\beta + (2\zeta_h\beta+1)(j\Phi_h)^2 + \left(\beta+\frac{\theta^2}{kC_p}\beta+2\zeta_h\right)(j\Phi_h)+1 = d_M(j\omega)$$

Where *det M* # 0, equation (11) can be writing by:

$$MM^{-1}\begin{bmatrix}Y(\omega)\\ V(\omega)\end{bmatrix} = M^{-1}\begin{bmatrix}\Phi_h^2\, Y_b(\omega)\\ 0\end{bmatrix} = \frac{1}{d_M}\begin{bmatrix}(j\Phi_h\beta+1) & \frac{\theta}{k}\\ -j\Phi_h\frac{\beta\theta}{C_p} & (1-\Phi_h^2)+2j\zeta_h\Phi_h\end{bmatrix}\begin{bmatrix}\Phi_h^2\, Y_b(\omega)\\ 0\end{bmatrix}$$

$$\begin{cases}Y(\omega) = (j\Phi_h\beta+1)\Phi_h^2\, Y_b(\omega)/d_M\\ V(\omega) = -j\Phi_h^3\frac{\beta\theta}{C_p}\, Y_b(\omega)/d_M\end{cases}$$

$Y_b(\omega)$ can give by:

$$Y_b(\omega) = -\frac{C_p d_M}{j\Phi_h^3\beta\theta}\, V(\omega)$$

Taking the superposition of the first three vibration modes of the bridge as solution, the equation (12) becomes:

$$-\frac{C_p d_M}{j\Phi_h^3\beta\theta}\, V(\omega) = \sum_{n=1}^{3}\frac{1}{(-\omega^2+2j\omega\zeta_n\omega_n+\omega_n^2)}\frac{1}{\sqrt{2\pi}}\frac{2P}{l\mu}\frac{1}{2j} * \left(\frac{e^{j(n\phi-\omega)T}}{j(n\phi-\omega)}+\frac{e^{j(n\phi-\omega)T}}{j(n\phi+\omega)}-\frac{1}{j(n\phi-\omega)}+\frac{1}{j(n\phi+\omega)}\right) sin\left(\frac{n\pi x}{l}\right) \quad (13)$$

4.2 Recovered energy

The energy obtained here is the power output giving by the formula:

$$|P| = \frac{|V|^2}{R}$$

- *V* is the voltage obtained in the piezoelectric element with equation (13). It is expressed by:

$$V(\omega) = -\frac{j\Phi_h^3\beta\theta}{C_p d_M}\sum_{n=1}^{3}\frac{1}{(-\omega^2+2j\omega\zeta_n\omega_n+\omega_n^2)}\frac{1}{\sqrt{2\pi}}\frac{2P}{l\mu}\frac{1}{2j} * \left(\frac{e^{j(n\phi-\omega)T}}{j(n\phi-\omega)}+\frac{e^{j(n\phi-\omega)T}}{j(n\phi+\omega)}-\frac{1}{j(n\phi-\omega)}+\frac{1}{j(n\phi+\omega)}\right) sin\left(\frac{n\pi x}{l}\right) \quad (14)$$

- *R* is a resistor set on the power harvesting circuit.

5. APPLICATION

5.1 Data examples

The bridge is considered a under a moving load of 30000 N. Table 1 shows several bridge parameters.

For the harvester, the characteristics of a PVDF are adopted and listed in Table 2 [9]. The resistance of the power harvester circuit has been set at 50 kΩ.

Table - 1: Bridge parameters [8]

Bridge parameters	values
Mass per unit length, μ	4406.78 *kg/m*
Bending stiffness, EJ	106 GN/m^2

Table - 2: PVDF characteristics [9]

Properties	Symbol	Values	Units
Young's modulus	Y_{33}	0.9	10^9 N/m^2
Stiffness	c_{33}	1.05	10^9 *Pa*
Piezoelectric charge constant	d_{33}	-34	10^{-12} *C/N*
Piezoelectric stress constant	e_{33}	35.7	10^{-3} C/m^2
Area of piezoelectric element	A	0,2 x 0,1	m^2
Thickness	h	515	10^{-6} *m*
Capacitance	C_p	2.6	10^{-9} *F*
Damping ratio	ζ	0,15	
Density	ρ	1470	kg/m^3
Relative dielectric constant	k	7.6	
Frequency	ω_h	22	*rad/s*

5.1 Results

The use of these parameters values in equation (14) can give the power output as a function of the position of the harvester, the velocity of the vehicle and the length of the bridge. Four types of bridge with a various vehicle speeds are studied. The results are shown in Table 3.

Table - 3: Table of power with parameters considered

Speed (km/h)	Maximum power $\frac{V^2}{R}, (10^{-6}W)$ For l = 25 m			
	x=1/4	x=1/2	x=2/3	x=3/4
36	49.951	99.915	75.004	49.951
54	47.030	94.053	70.602	47.021
72	46.940	93.867	70.479	46.946
90	48.005	96.010	72.105	48.036

Speed (km/h)	Maximum power $\frac{V^2}{R}, (10^{-6}W)$ For l = 30 m			
	x=1/4	x=1/2	x=2/3	x=3/4
36	69.886	139.77	104.83	69.886
54	74.476	148.95	111.71	74.476
72	74.244	148.49	111.37	74.244
90	71.085	142.17	106.63	71.085

Speed (km/h)	Maximum power $\frac{V^2}{R}, (10^{-6}W)$ For l = 35 m			
	x=1/4	x=1/2	x=2/3	x=3/4
36	101.18	202.36	151.82	96.638
54	93.938	187.88	140.96	89.723
72	106.05	212.11	159.14	101.30
90	101.26	202.52	151.95	96.719

Speed (km/h)	Maximum power $\frac{V^2}{R}, (10^{-6}W)$ For l = 50 m			
	x=1/4	x=1/2	x=2/3	x=3/4
36	187.83	375.61	281.82	187.86
54	197.41	394.90	296.41	197.63
72	192.57	384.86	288.44	192.15
90	218.97	438.20	329.03	219.41

Figure 4 and Figure 5 shows a voltage obtained, a maximum power and the displacement plots of the corresponding bridge. The example considered is in the quarter and half position of bridge. His length is l = 50 m and the vehicle speed is 90 km/h.

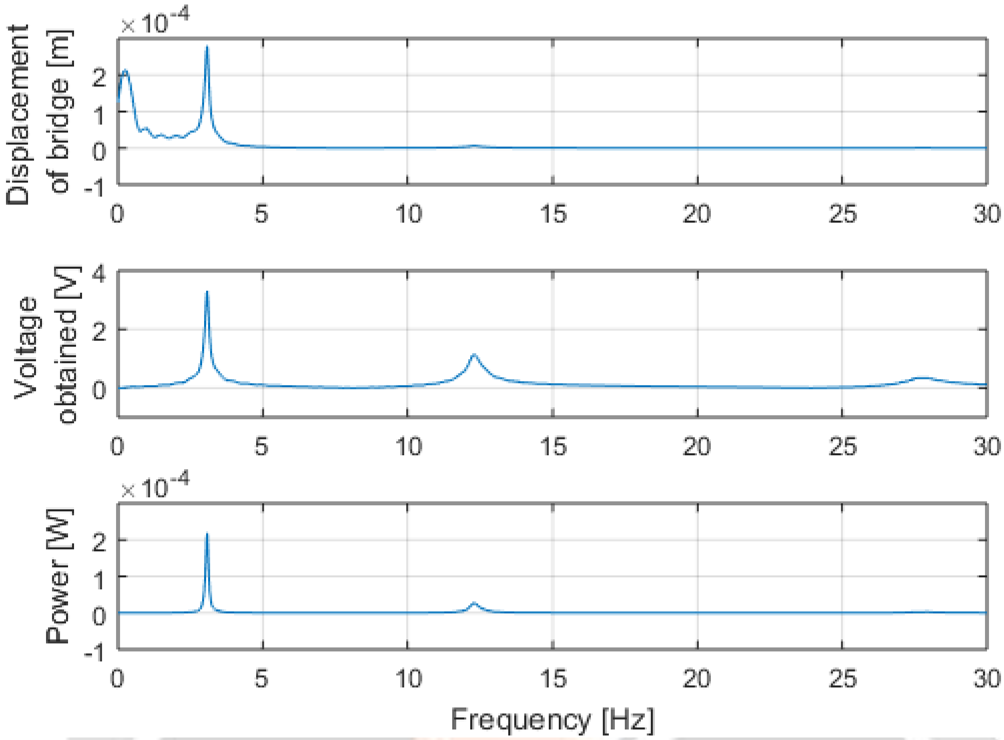

Fig - 4: Displacement of bridge, voltage obtained and power output

with *l = 50 m, x= 12.5 m* and *v = 90 km/h*.

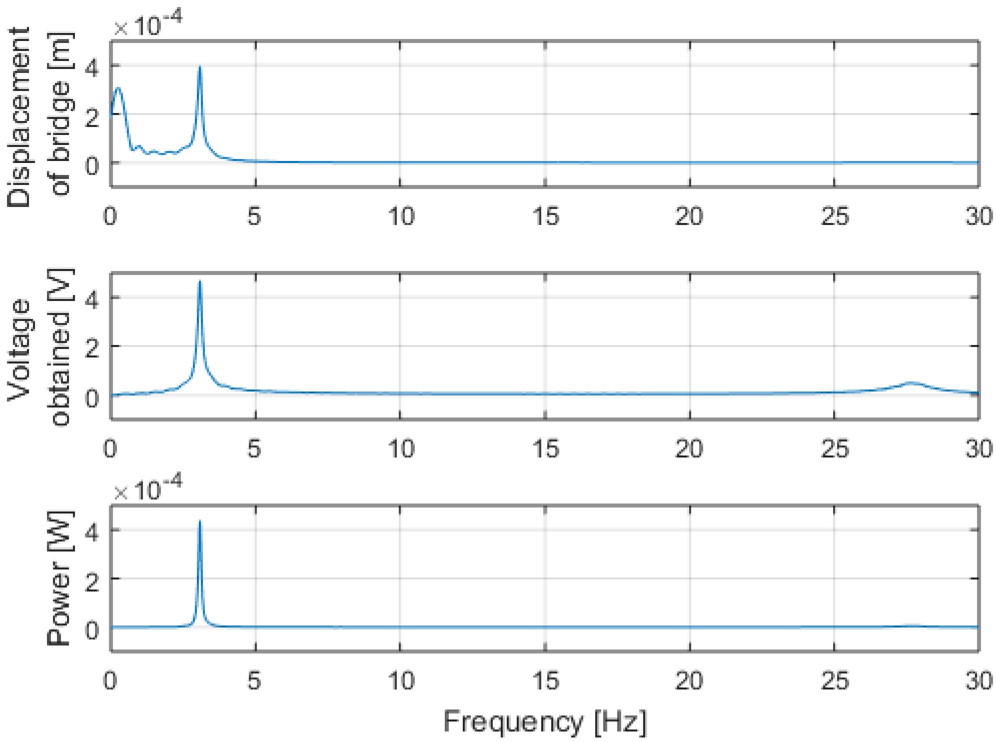

Fig - 5: Displacement of bridge, voltage obtained and power output

with *l = 50 m, x= 25 m* and *v = 90 km/h*.

5.2 Interpretation of results

From the obtained results, it is found that:

- The power output is the best in the half length of the bridge regardless of the length and speed of the vehicle.
- The power value increases as the length of the bridge increases. It means that the power is proportional to the speed.
- On the one hand, a shorter bridge is more promising with a slower speed and on the other hand, a longer bridge gives more energy with a faster speed. Example for the length l = 25 m, the maximum power of *99.9 µW* is obtained at a speed *c = 36 km/h*, and for the length *l = 50 m*, the maximum power is *438.20 µW* with a speed *c = 90 km/h.*

5. CONCLUSION

Piezoelectric energy harvester from bridge vibrations caused by a constant moving load is shown in this work. The response of bridge under moving load and the piezoelectric harvester are studied separately in frequency domain. Then the two models are combined to make up a harvesting system. Mathematical resolutions of this model in frequency domain show that an electric energy can be obtained. Numerical application is done to know the order of power output. For the displacement of bridge, the superposition of the first three mode of vibration is choosing. The voltage obtained in power harvester circuit with a fixe resistor 50 KΩ is 2,2351 at 4,6808 volts. Which means that the power output is between 99,9 µW and 438.20 µW. This energy can be processed and storage to power a LED traffic light in the bridge for future works.

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