Numerical Study of Differential Difference equation Using Hermite Collocation Method

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ABSTRACT

Present paper deals with the study of one dimensional advection diffusion equation involving non linear terms. Problem is solved using Hermite collocation method which is a combination of orthogonal collocation method and Hermite interpolating polynomials. Numerical results have been presented in terms of time space graphs. Values obtained from HCM are compared with OCM and pdepe solver.

Keywords: *Cubic Hermite Polynomials, Collocation points, Hermite collocation method*.

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1. INTRODUCTION

Advection diffusion or reaction problems deals with the class of heat and mass transfer problems encountered in various real life problems. These problems are mainly encountered in physical systems involving convective transport of chemical species such as adsorption, dying, bleaching, and washing etc. Construction of mass transfer problems lead to the development of a mass transfer problem in the form of ordinary or partial differential equations involving constraints in the form of initial and boundary conditions.

Present study deals with the solution of such type of two point boundary value problems using the technique of Hermite collocation method (HCM) (Akgonullu et al 2011, Gulsu et al. 2011 & Parand et al. 2010**)**. It is the combination of orthogonal collocation method and Hermite interpolating polynomials.It belongs to the class of weighted residual methods and variational principle. This technique has the property to convert mixed collocation into interior collocation method. Instead of complicated tri-diagonal matrix structure, HCM involves block diagonal structure of coefficient matrix of order (2N×2N) and thus reduces the number of equations.HCM yields greater accuracy for a smooth solution with few equations at each time level. The choice of Hermite interpolating polynomials is far better than that of Lagrangian interpolation polynomials due to its approximation at tangent points of the curve which does not require additional condition of continuity.

2. Hermite Collocation Method

In Hermite interpolating polynomials, as discussed earlier both the function and its derivative are to be assigned values at interpolating point. An nth-order Hermite polynomial in *x* is a polynomial of order 2n+1 and therefore, cubic Hermite interpolating polynomial is a particular case of general Hermite interpolating polynomial. It consists of two node points and is defined as:

$$H_1(x) = 2x^3 - 3x^2 + 1 \quad H_2(x) = x^3 - 2x^2 + x$$

$$H_3(x) = -2x^3 + 3x^2 \quad H_4(x) = x^3 - x^2$$

Figure1 shows the behavior of the cubic Hermite interpolating polynomial. The values of H_1,H_2, H_3 and H_4 lies within [0, 1] as *x* goes from 0 to 1 and their derivatives are unity or zero at the end points.

2.1 Collocation Points

Jacobi Polynomials is a well known family of hyper-geometric polynomials. Legendre as well as Chebyshev polynomials are special case of Jacobi polynomials. The roots of the Jacobi polynomial have been used as collocation points over the normalized interval of [0,1].The explicit form of Jacobi polynomials that has been used [Andrews 1962,Arora et. al.2005] is

$$P_k^{(\alpha,\beta)}(x) = \sum_{m=0}^{k} \frac{(-1)^{k-m}(1+\beta)_k(1+\alpha+\beta)_{k+m}}{m!(k-m)!(1+\beta)_m(1+\alpha+\beta)_k}\left(\frac{1+x}{2}\right)^m$$

Where α and β are parameters.

2.2 Stability Analysis

In HCM, two collocation points have been taken within each element. The residual is set equal to zero at the collocation points and otherwise nonzero at non collocation points. Therefore to achieve stability in the numerical results, number of elements should be increased instead of number of collocation points. It signifies the fact that in case of HCM, stability and convergence depends upon number of elements.For this purpose L_2 and L_∞ norms defined below have been calculated

$$\|y\|_2 = \sqrt{h\sum_{j=0}^{N}\left|\left(y(x_j,t) - y^{\wedge}(x_j,t)\right)^2\right|}$$

$$\| y \|_\infty = \underset{x_j \varepsilon [0,1]}{Max}\left|\left(y(x_j,t) - y^{\wedge}(x_j,t)\right)\right| \quad ; j=1,2,..N$$

Where y is the analytic value and $y^{\wedge}$ is the numerical value calculated using HCM.

Algorithm of HCM

To solve the problems using HCM, following algorithm should be followed

1) Expand the trial function in term of cubic Hermite basis functions with unknown coefficients.
2) Make the trial solution satisfy the boundary and initial conditions.
3) Substitute the trial function into the given problem to get residual.
4) Set the residual to zero at collocation points.
5) Solve the resulting system of equations using MATLAB solver.

3. Problem

Consider two point boundary value problem:

$$\frac{\partial y}{\partial t} = \frac{1}{P}\frac{\partial^2 y}{\partial x^2} - \frac{\partial y}{\partial x} - y \qquad (x,t)\in(0,1)\times(0,T) \tag{1}$$

$$y - \frac{1}{P}\frac{\partial y}{\partial x} = 0 \qquad \text{at } x = 0, \quad \text{for all } t \geq 0 \tag{2}$$

$$\frac{\partial y}{\partial x} = 0 \qquad \text{at } x = 1, \quad \text{for all } t \geq 0 \tag{3}$$

$y = 1$ at t= 0, for all x (4)

The problem has been solved by HCM for different values P with respect to t as well as x space. To discretize the problem 140 elements with two interior collocation points have been taken in each element. The analytic solution involves the complicated terms of complimentary error functions which have been solved during the series reported by Andrews 1962. In figure 2, the behavior of solution profiles for numerical and analytic values is checked. The numerical and analytic values are compared for P = 4, 16, 32 and 128. It is observed from this figure that area under the curve increases with the increase in the value of P and converges to steady state condition more rapidly for large values of P as compared to small values of P. It is also clear from this figure that analytic and numeric values are matching to a desired limit and numerical values are also converging to steady state condition smoothly.

In figure 3, numerical results are given in the form of breakthrough curves for various values of P ranging from 1 to 8 at t=0.1. It is evident from this figure that with the increase in the value of P= 8, the curves approaches to zero more rapidly as compared to small values of P= 1 or 4. Thus, desired numerical results may be achieved for large values of P.

In figure 4, the solution profiles follow Gaussian shape for various values of t for P=4. It is obvious from this figure that for t= 0.1, the curves converge to zero more rapidly as compared to t= 0.2 or 0.4.

Figure 5, the effect of Legendre and Chebyshev polynomials at x = 1 is checked. It is found that both the polynomials give results upto the desired accuracy, however the accuracy is higher in case of Legendre polynomial as compared to Chebyshev Polynomial. Hence, Legendre polynomial gives better numerical results as compared to Chebyshev polynomials.

Figure 6, a comparison between HCM, OCM and *pdepe* solver has been presented for P=5. In this figure it is evident that relative error is very high for OCM and *pdepe* solver as compared to HCM. As time increases, the behavior of solution profile is very abrupt. In case of HCM, the relative error is very small and is less than 1 % which goes upto more than 100 and 50 times in case of OCM and *pdepe* solver, respectively.

In Figure 7, $||y||_2$ norms have been presented for P varying from 1 to 16 with respect to time. The number of elements has been taken 140. The variation in $||y||_2$ norm curves for P= 1 or 2 is not a great deal however for P = 1 or 2, $||y||_2$ approaches to zero more rapidly as compared to P = 4 or 16. Therefore $||y||_2$ approaches to zero for small value of P.

In Figure 8, $||y||_\infty$ norms have been presented for P varying from 1 to 16 with respect to time. The graphs of $||y||_\infty$ norm are overlapping for P = 1 and P = 2. For P = 1 or 2, $||y||_\infty$ approaches to zero more rapidly as compared to P = 4 or 16. Hence stable numerical results can be attained for small values of P.

In Figure 4.25, a comparison between $||y||_\infty$ and $||y||_2$ has been presented for P = 8. It is evident that both $||y||_\infty$ and $||y||_2$ provide error in exponential form and the values of $||y||_2$ norm converges to zero more rapidly as compared with $||y||_\infty$. Hence convergence is found to be faster in case of $||y||_2$ as compared to $||y||_\infty$.

Comparison of $||y||_\infty$ and $||y||_2$ norms for various values of P has been shown in table.

4. Conclusion:

A numerical scheme based on HCM has been presented for the solution of boundary value problem. From these figures and tables, it is observed that the accuracy of the method is of order (h^2). It is also observed that Legendre polynomial gives better numerical results as compared to Chebyshev polynomials. Numerical results demonstrate the stability and accuracy of the method for any value of P. This method is easy to implement and yields very accurate results.

5. Reference:

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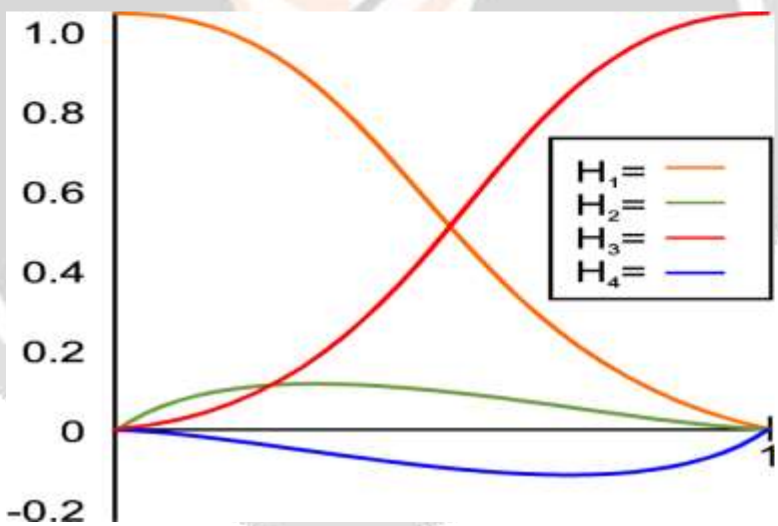

Figure 1. Cubic Hermite interpolating polynomials.

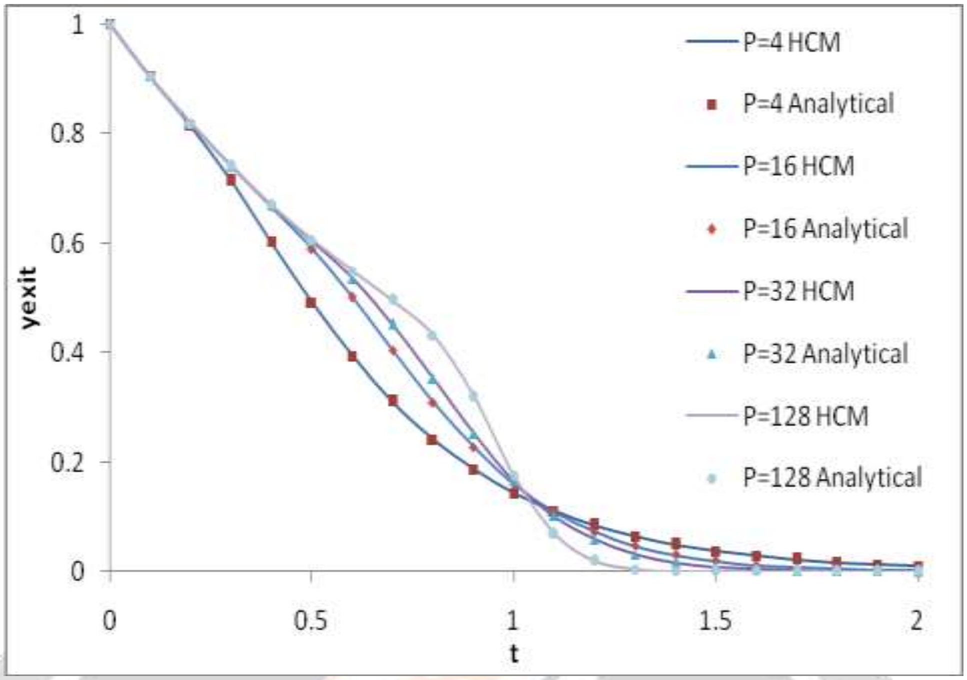

Figure 2: Comparison of analytic and numerical values for different values of P.

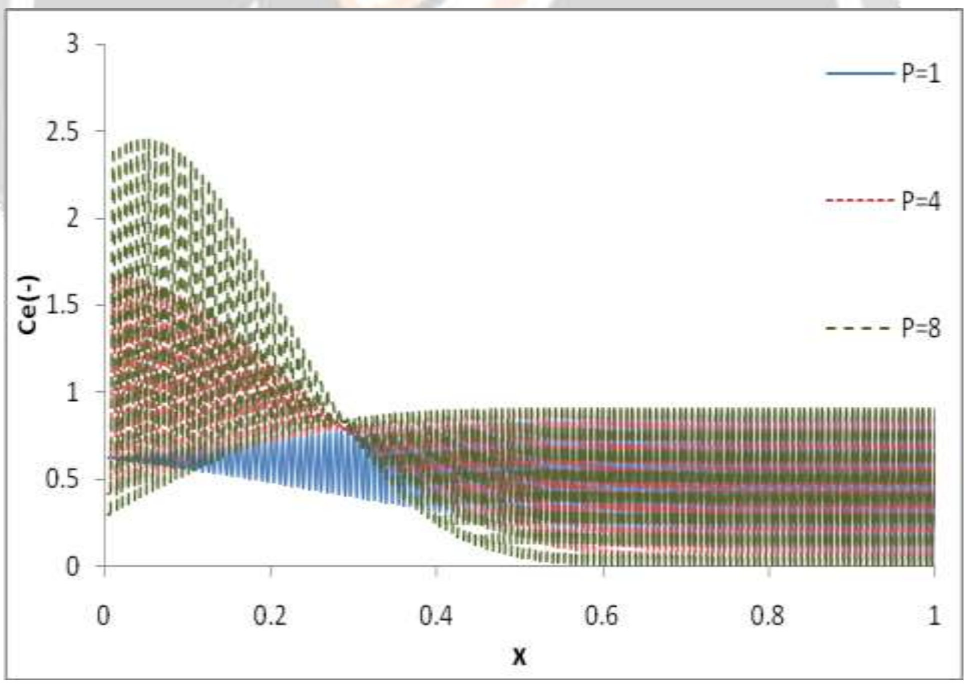

Figure 4.19: Solution profiles for different values of P at t=0.1.

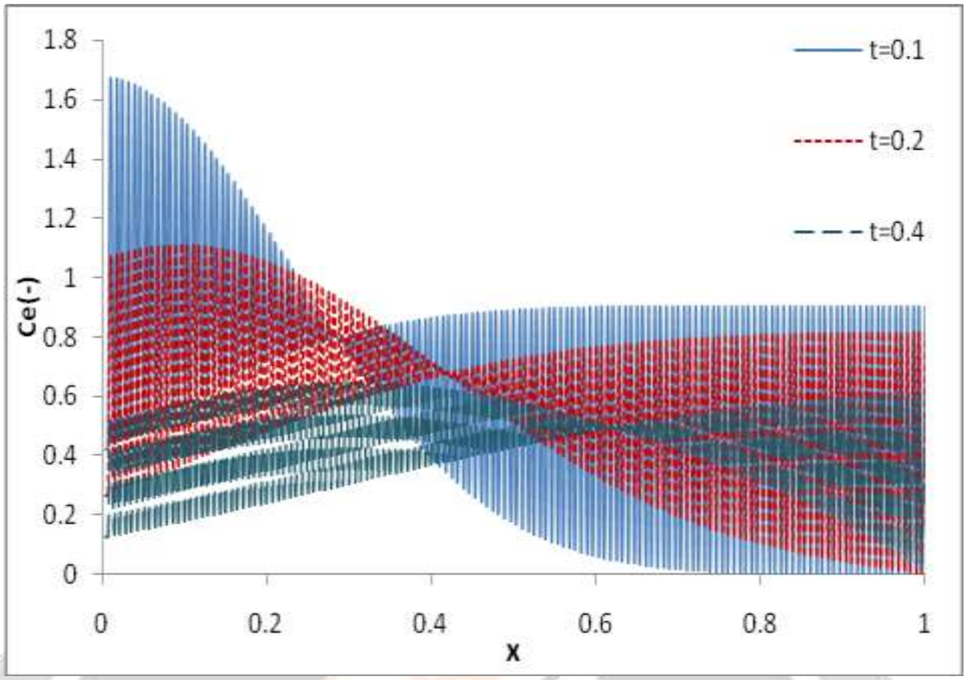

Figure 4.20: Behavior of solution profiles for different values of t.

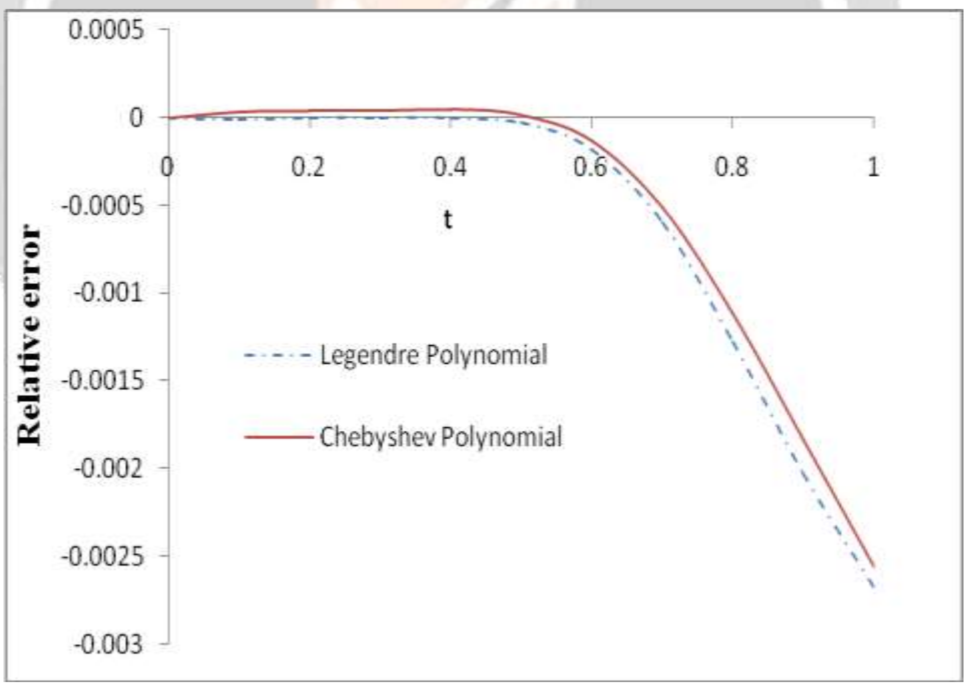

Figure 5: Comparison of different polynomials for P = 40.

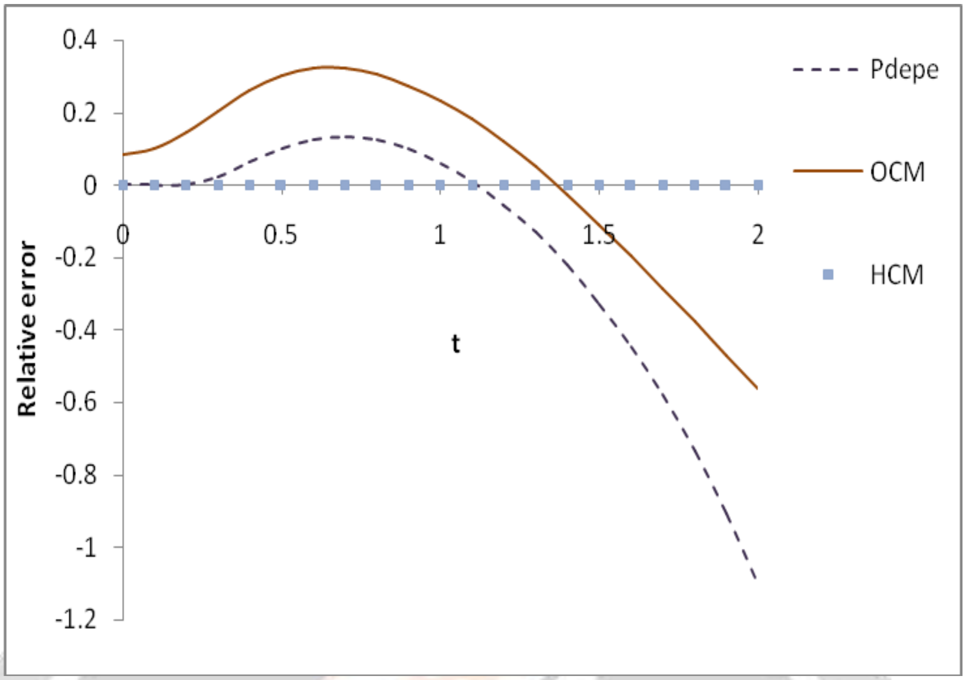

Figure 6: Comparison of OCM, HCM and Pdepe.

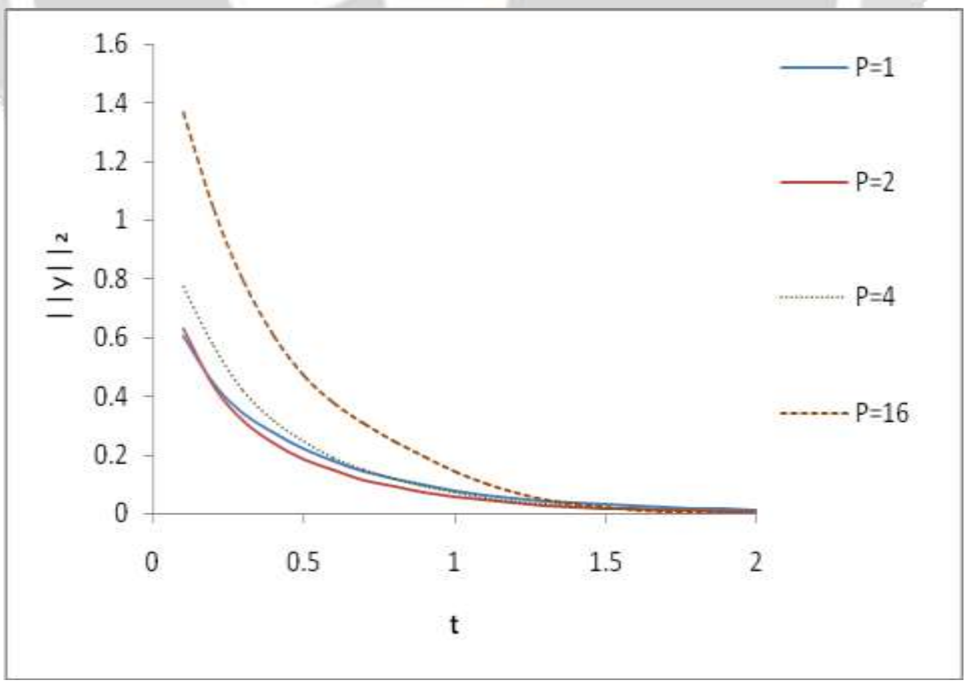

Figure 7: Comparison of $||y||_2$ norm for various values of P

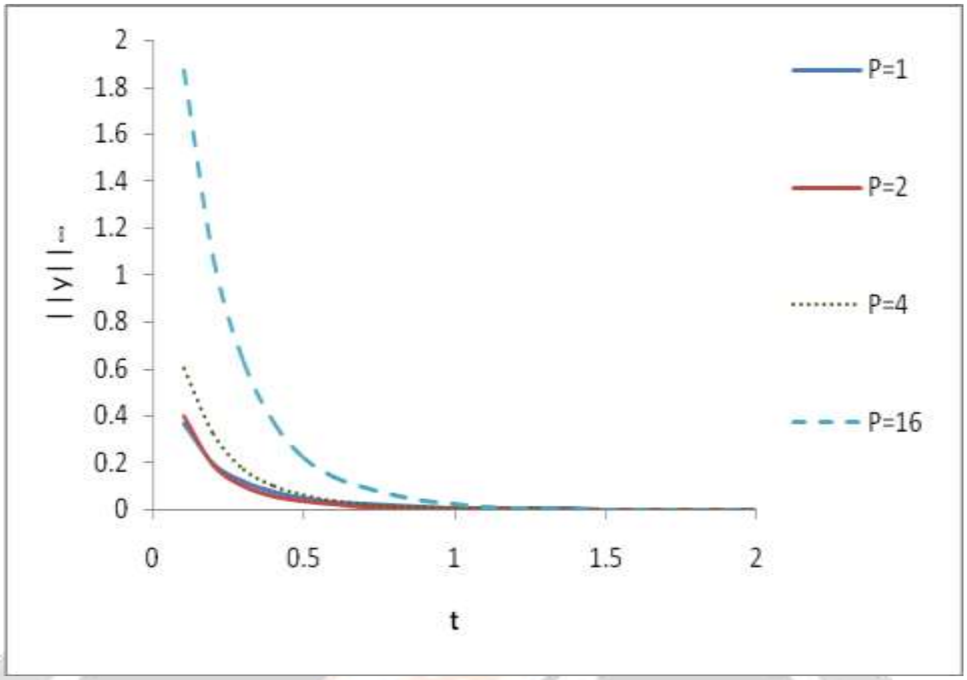

Figure 8: Comparison of $||y||_\infty$ norm for various values of P

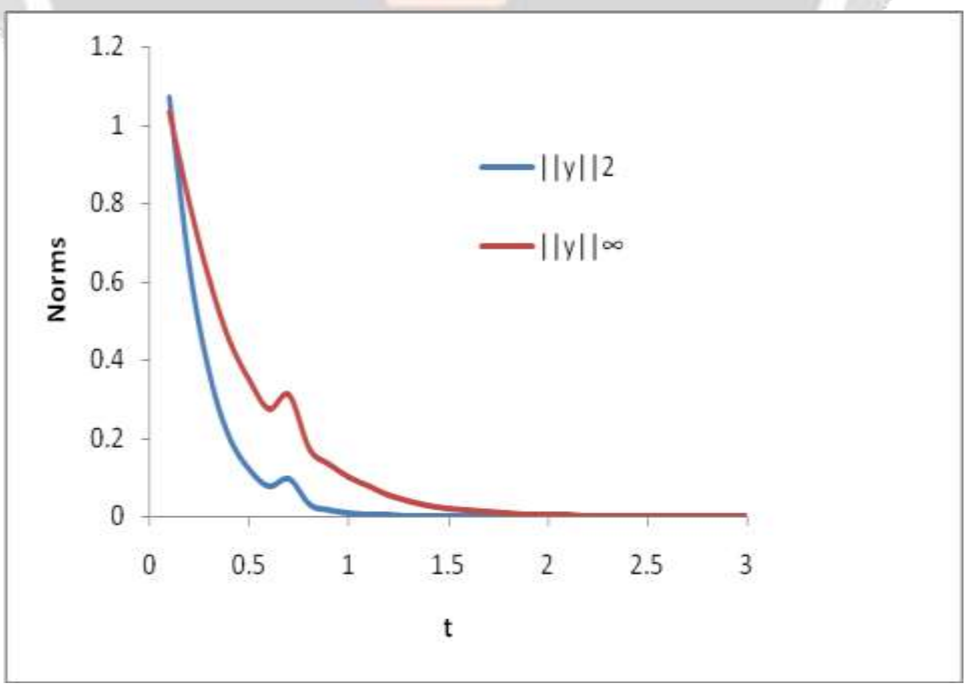

Figure 9: Comparison of $||y||_\infty$ and $||y||_2$ norms for P=8.

Comparison of $||y||_\infty$ and $||y||_2$ norms for different values of P

t	P=1,$\|\|y\|\|_\infty$	P=1,$\|\|y\|\|_2$	P=2,$\|\|y\|\|_\infty$	P=2,$\|\|y\|\|_2$	P=4,$\|\|y\|\|_\infty$	P=4,$\|\|y\|\|_2$	P=16,$\|\|y\|\|_\infty$	P=16,$\|\|y\|\|_2$
0.1	0.36631271	0.605237733	0.39648	0.6296663	0.5993787	0.7741955	1.86883156	1.36705214
0.5	0.04835257	0.219892184	0.034645	0.1861312	0.05914429	0.243196	0.21981015	0.46883915
1	0.00607911	0.077968671	0.003191	0.0564881	0.00487736	0.0698381	0.02024726	0.14229288
1.5	0.0008591	0.02931048	0.00031	0.0175942	0.00033454	0.0182904	0.00038515	0.01962519
2	0.00012854	0.011337762	3.11E-05	0.0055738	2.1772E-05	0.0046661	3.1809E-06	0.00178349
2.5	1.9039E-05	0.004363394	3.15E-06	0.0017736	1.4115E-06	0.0011881	1.9027E-08	0.00013794
3	2.5969E-06	0.001611476	3.07E-07	0.0005543	9.2789E-08	0.0003046	9.921E-11	9.9604E-06