ON THE GEOMETRY OF GENERALIZED WEINGARTEN HYPERSURFACES

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ABSTRACT

In this paper we derive a condition of transversality of two given hypersurfaces in pseudo-Riemnnaian manifolds, along its boundary. This condition is given by the ellipticciy of the Newton transformations.

Keyword : *Newton transformations, Symmetric functions, Transversality.*

1. INTRODUCTION

Let M^{n+1} an $(n+1)-$ dimentional connected Riemannian manifold and M^n be a closed hypersurface embedded in M^{n+1}. Denoting by $x_1, \dots, x_n$ its principal curvatures.

For $1 \leq k \leq n-1$, we define the $k-$ mean curvature H_k of M^n by

$$\binom{n}{k} H_k = \sigma_r(\,x_1, \dots, x_n).$$

Where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and $\sigma_r : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are the elementary symmetric functions define by

$$\sigma_r(\,x_1, \dots, x_n) = \sum_{i_1 < \cdots < i_k} x_{i_1} \dots x_{i_k}.$$

For instance, $H_1 = H$ is the mean curvature, H_2 is, up a constant, the scalar curvature and H_n is the Gauss curvature.

The Alexandrov's sphere theorem [1] states that the round sphere is the only closed hypersurface embedded in $\mathbb{R}^{n+1}$.

This result is not true for the case of immersed (and non embedded) hypersurfaces [16,10].

Ros [13] later prove that the above result is true for hypersurfaces of constant H_k for k>1, embedded in Euledean space. The result was generalized by Montiel and Ros [11] for hypersurfaces with constant H_k embedded in $\mathbb{H}^{n+1}$ and $\mathbb{S}_+^{n+1}$.

Koh [8] and Koh-Lee [9] later gived an analogue for the case of constant mean curvature ratio $\frac{H_k}{H_1}$ hypersurfaces.

In a recent work de Lima [5] gived a generalization of the Alexandrov theorem for the case of linear Weingarten hypersurfaces embedded in Euclidean space. That is an hypersurface where H_k and H are linearly related. this means that for a and $b > 0$, $H_k = aH + b$.

In this work we consider a compact generalized Weingarten hypersurfaces (or $(r,s)-$ Weingarten hypersurface) embedded in $\mathbb{R}^{n+1}$. That is an hypersurface whose some of the $k-$mean curvatures H_k are lineary related. ie : for $0 \leq s \leq r \leq n$, the relation

$$a_s H_s + \cdots . + a_r H_r = b.$$

holds, where $b > 0$ and $(a_s, \ldots, a_r) \neq (0, \ldots, 0)$.

We prove the following result:

THEOREM

Let M^n be a closed, oriented $(r,s)-$Weingarten hypersurface embedded in $\mathbb{R}^{n+1}$ with non vanishing $k-$mean curvature H_k.

If we have one of the following cases:

i. *For some integers r and s satisfying the inequality $0 \leq s \leq r \leq n-1$, the following linear relation*

$$a_s H_s + \cdots . + a_r H_r = b,$$

holds, where $b > 0$ and a_i with $(a_s, \ldots, a_r) \neq (0, \ldots, 0)$.

ii. *For some integer r where $0 \leq r \leq n-1$, the relation*

$$H_r = a_1 H_1 + \cdots + a_{r-1} H_{r-1},$$

holds, with $(a_1, \ldots, a_{r-1}) \neq (0, \ldots, 0)$.

Then M^n is the geodesic hypersphere.

2. MAIN RESULTS

The main result in this work is:

THEOREM 1.

Let M^n be a closed, oriented $(r,s)-$Weingarten hypersurface embedded in $\mathbb{R}^{n+1}$ with non vanishing $k-$mean curvature H_k.

If we have one of the following cases :

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$$H_r = a_1 H_1 + \cdots + a_{r-1} H_{r-1},$$

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Then M^n is the geodesic hypersphere.

PROOF.

$\varphi : M^n \longrightarrow \mathbb{R}^{n+1}$ be an $n-$dimentional closed hypersurface embedded in $\mathbb{R}^{n+1}$. Denoting by N the unit vector field normal to M^n.

i. Since M^n is a compact and embedded in $\mathbb{R}^{n+1}$, it bounds a domain Ω in $\mathbb{R}^{n+1}, \partial\Omega = M^n$.

Moreover, always by compactness, M^n has at least an elliptic point, that is a point of M^n, where all the principal curvatures are positive. This imply that all H_r are positive functions.

For $1 \leq i \leq n$, the Minkoswki formula is written as (See [6])

$$\int_{M^n} H_{i-1} dM + \int_{M^n} H_i \langle \varphi, N \rangle dM = 0.$$

So

$$\sum_{i=}^{r} a_i \int_{M^n} H_{i-1} dM = -\sum_{i=}^{r} a_i \int_{M^n} H_i \langle \varphi, N \rangle dM = -\int_{M^n} b \langle \varphi, N \rangle dM.$$

On the other hand, since H_1 is strictly positif and by the inequality (See [11]) :

$$H_{k-1}.H_l \geq H_k.H_{l-1},$$

we obtain

$$\sum_{i=}^{r} a_i \int_{M^n} H_{i-1} dM = \sum_{i=}^{r} a_i \int_{M^n} H_{i-1} \frac{H_1}{H_1} dM,$$

$$\geq \sum_{i=}^{r} a_i \int_{M^n} H_i \frac{1}{H_1} dM,$$

$$\geq b \int_{M^n} \frac{1}{H_1} dM,$$

$$\geq b(n+1) vol\Omega.$$

Where the last inequality follows from theorem 1 in [13].

On the other hand, by applying the divergence theorem, it is not difficult to see that

$$-\int_{M^n} b \langle \varphi, N \rangle dM = b(n+1) vol\Omega.$$

This imply that all the above inequalities are equals. In particular we obtain :

$$\int_{M^n} \frac{1}{H_1} dM = (n+1) vol\Omega.$$

Wich implies that M^n is the round sphere (See [6]).

ii. For $1 \leq r \leq n$ we have

$$\int_{M^n} H_{r-1} dM + \int_{M^n} H_r \langle \varphi, N \rangle dM = 0.$$

So,

$$\int_{M^n} H_{r-1} dM = -\int_{M^n} H_r \langle \varphi, N \rangle dM,$$

$$= -\sum_{i=1}^{r-1} a_i \int_{M^n} H_i \langle \varphi, N \rangle dM,$$

$$= \sum_{i=1}^{r-1} a_i \int_{M^n} H_{i-1} \langle \varphi, N \rangle dM.$$

Thus

$$\int_{M^n} \left(H_{r-1} - \sum_{i=1}^{r-1} a_i H_{i-1} \langle \varphi, N \rangle \right) dM = 0.$$

This gives

$$H_{r-1} = \sum_{i=1}^{r-1} a_i H_{i-1} \langle \varphi, N \rangle.$$

and by a reccursive argument, we obtain

$$H_1 = C.H_0 = C.$$

Were C is a constant depend only on $a_1, \dots, a_{r-1}$

Hence M^n is the round sphere.

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