OSCILLATION BEHAVIOR OF SOLUTIONS OF NEUTRAL DIFFERENTIAL EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

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ABSTRACT

we study the oscillatory behaviour of neutral differential equation. we obtain for the new oscillation of neutral differential equations with positive and negative coefficients. In fact ,every solution of the neutral differential equation is oscillates, while the equation has an eventually positive solution. otherwise, the equation is non oscillates, while the equation has an eventually negative solution.

Keywords: *Oscillation, Neutral differential equation, positive and negative coefficients, eventually positive, eventually negative, bounded, unbounded.*

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1. INTRODUCTION:

In this paper, We discuss the oscillatory behaviour of all solutions of the following neutral differential equation with positive and negative coefficient

$$[u(k)-G(k)u(k-a)\,]'+E(k)u(k-\omega)-F(k)u(k-\rho)=g(k), k\geq k_0 \quad (1.1)$$

where

$E,F,G \in C\ ([k_0,\infty),\mathbb{R}^+), f\in C([k_0,\infty),\mathbb{R})\quad a>0, \omega\geq 0, \rho\geq 0.$

If $g(x)\equiv 0$, then the above equation becomes

$$[u(k)-G(k)u(k-a)]'+E(k)u(k-\omega)-F(k)u(k-\rho)=0\,,\quad k\geq k_0 \quad (1.2)$$

The oscillation of every solutions of equations has been investigated by several authors. In particular, all known oscillation results for equation (1.2) require the following condition:

$$\int_{k_0}^{\infty}[E(t) - F(t-\omega+\rho)]ds = \infty \quad (1.3)$$

which has played a very important role in the study of (1.2). However, we obtained some new sufficient conditions for the oscillation of equations respectively, which do not require the condition(1.3).They used the following known condition

$$G(k) + \int_{k-\omega+\rho}^{k} F(t)dt \equiv 1 \qquad (1.4)$$

In this paper, our aim is to give some new sufficient conditions for the oscillation of neutral differential equation without the usual condition. Our results improve the known results in the literature. we remark that the oscillation behaviours of the solutions of neutral differential equations. In fact, every solution of (1.2) oscillates, while equation (1.1) has an eventually positive solution. Moreover $u(k)$ is an eventually positive solution of (1.2) and implies that $-u(k)$ is an eventually negative solution of (1.2).This property has been extensively used in the study of the oscillatory behaviours of differential equations without forced terms. But ,one can't say that $-u(k)$ is an eventually negative solutions of (1.1) when $u(k)$ is an eventually positive solution of (1.2).

Let $l = max\{a, \omega\}$; by a solution of equation (1.1), we mean a function $u(k) \in C([k_1 - l), \infty, \mathbb{R})$ for some $k \geq k_1$ which satisfies equation (1.1) for $k \geq k_1$. We recall that a nontrivial solution of equation (1.1) is said to be oscillatory if it has arbitrarily large zeros. Otherwise, the solution is called non oscillatory.

For any real number $m \in [0, \omega - \rho]$ let

$$W_m(k) = G(k) + \int_{k-m}^{k} F(t)dt + \int_{k}^{k-m+\omega-\rho} E(t)\, dt \qquad (1.5)$$

If $m = 0$, then the equation becomes

$$W_o(k) = G(k) + \int_{k}^{k+\omega-\rho} E(t)dt.$$

and If $m = \omega - \rho$, then the equation becomes,

$$W_{\omega-\rho}(k) = G(k) + \int_{k-\omega+\rho}^{k} F(t)dt.$$

Throughout this paper, we let

$$\rho_m = \begin{cases} a, & when\ G(k) \equiv 0\ and\ m = \omega - \rho \\ max\{a, \omega\}, & otherwise \end{cases}$$

$$\gamma_m = \begin{cases} a, & when\ G(k) \equiv 0\ and\ m = \omega - \rho \\ max\{a, \rho\}, & otherwise \end{cases}$$

and the following conditions holds:

$$\begin{cases} D(k) = E(k) - F(k-\omega+\rho) \geq 0, \\ \left|\int_{k_0}^{\infty} g(t)dt\right| < \infty \end{cases} \qquad \text{(C)} \qquad \text{and identically not zero.}$$

2. PRELIMINARIES:

2.1 Eventually positive

A real valued function $y(x)$ defined on an interval $[r_y,\infty)$ is called eventually positive if $y(r)>0$ on $[r,\infty)$ for some $\tilde{r}>r_y$.

2.2 Eventually negative

A real valued function $y(x)$ defined on an interval $[r_y,\infty)$ is called eventually negative if $y(r)<0$ on $[r,\infty)$ for some $\tilde{r}>r_y$.

1.1.7 Oscillatory

A non-trivial solution y is said to be oscillatory if it has arbitrary large zeros for $t\geq t_0$ that is there exists a sequence of zero $\{t_n\}$ of y. i.e., $y(t)=0$ such that $\lim_{n\to\infty} t_n=\infty$. Otherwise, the solution is said to be non oscillatory.

3. MAIN RESULTS:

Theorem 3.1 :

Suppose there exists a number $k\in[0,\omega-\rho]$ such that

$Y_m(k)=G(k)+\int_{k-m}^{k}F(t)dt+\int_{k}^{k-m+\omega-\rho}E(t)dt\leq 1$ holds eventually. suppose further that $lim_{n\to\infty}\inf k\int_k^\infty[E(t+\omega-\rho)-F(t)]dt>\frac{max[a,\omega+\rho]}{4}$ Holds and that

$$G(k-\omega)[E(k)-F(k-\omega+\rho)]\geq[E(k-a)-F(k-\omega+\rho-a)] \qquad (3.1.1)$$

For large k.Then every solution of

$$[u(k)-G(k)u(k-a)]'+E(k)u(k-\omega)-F(k)u(k-\rho)\leq 0 \qquad (3.1.2)$$

oscillates.

Proof:

suppose to contrary that $u(k)$ is an eventually positive solution of (3.1.2).

Then, by using [3] lemma1, $w(k)>0$ and $w'(k)\leq 0$ for large k.

By using lemma1,we have

$$\begin{aligned} w'(k) &= -D(k-m+\omega-\rho)u(k-m-\rho)\\ &= -D(k-m+\omega-\rho)\big(w(k-m-\rho)+G(k-m-\rho)\,u(k-m-\rho-a)\big)\\ &\quad - D(k-m+\omega-\rho)\left(\int_{k-2m-\rho}^{k-m-\rho}F(t)u(t-\rho)dt\right.\\ &\quad \left.+\int_{k-m-\rho}^{k-2m+\omega-2\rho}E(t)u(t-\omega)dt\right). \end{aligned}$$

Hence

$$w'(k)+D(k-m+\omega-\rho)w(k-m-\rho)+G(k-m-\rho)$$

$$D(t-m+\omega-\rho)u(k-m-\rho-a) \le 0 \quad (3.1.3)$$

For large k. From

$$[u(k)-G(k)u(k-a)]' + E(k)u(k-\omega) - F(k)u(k-\rho) = 0,$$

we have

$$w'(k-a) + D(k-m+\omega-\rho-a)u(k-m-\rho-a) = 0.$$

By (3.1.3) and above inequality, we have

$$[w(k)-w(k-a)]' + D(k-m+\omega-\rho)u(k-m-\rho) \le \big(D(k-m+\omega-\rho-a) - G(k-m-\rho)D(k-m+\omega-\rho)\big)\ u(k-m-\rho-a) \le 0,$$

Which $w(k)$ is an eventually positive of the recurrence relation

$$[w(k)-w(k-a)]' + D(k-m+\omega-\rho)w(k-m-\rho) \le 0,$$

Which is a contradiction.

Hence ,

$u(k)$ is eventually positive solution and

$$[u(k)-G(k)u(k-a)]' + E(k)u(k-\omega) - F(k)u(k-\rho) \le 0$$

has a oscillatory solution.

Theorem 3.2:

Assume that condition

$$\begin{cases} D(k) = E(k) - F(k-\omega+\rho) \ge 0, \\ \left|\int_{k_0}^{\infty} g(t)dt\right| < \infty \end{cases}$$ holds. suppose there exists a real number $m \in [0, \omega-\rho]$ such that

$$w_m(k) = G(k) + \int_{k-m}^{k} F(t)dt + \int_{k}^{k-m+\omega-\rho} E(t)dt \le 1 \quad (3.2.1)$$

For large k. Let

$$w(k) = u(k) - G(k)u(k-a) - \int_{k-m}^{k} F(t)u(t-\rho)dt + \int_{k}^{k-m+\omega-\rho} E(t)u(t-\omega)dt + \int_{k}^{\infty} g(t)dt, \quad (3.2.2)$$

Then we have:

(i) If $x(t)$ is an eventually positive solution of

$$\left[u(k) - G(k)u(k-a) + \int_{k}^{\infty} g(t)dt\right]' + E(k)u(k-\omega) - F(k)u(k-\rho) \le 0, \quad (3.2.3)$$

then we have

$w'(k) \le 0$ and $w(k) > 0$ for large k. (3.2.4)

(ii) If $u(k)$ is an eventually negative solution of

$$\left[u(k) - G(k)u(k-a) + \int_k^\infty g(t)dt\right]' + E(k)u(k-\omega) - F(k)u(k-\rho) \geq 0, \quad (3.2.5)$$

Then we have:

$w'(k) \geq 0$ and $w(k) < 0$ for large k. (3.1.6)

Proof:

(i) Let $k_1 \geq k_0$ such that $u(k) > 0$ for $k \geq k_1 - k_0$.

Then from (3.2.3) and (3.2.2),we have

$$w'(k) \leq -D(k-m+\omega-\rho)u(k-m-\rho) \leq 0, \ k \geq k_1 \quad (3.2.7)$$

So, $w(k)$ is non increasing for $k \geq k_1$.

If $w(k) > 0$ does not hold, then eventually $w(k) < 0$,which implies that there exists a constant $\eta > 0$ and $k_2 \geq k_1$ such that $w(k) \leq -\eta$ for $k \geq k_2$.

By (3.2.2) we have

$$u(k) \leq -\eta + G(k)u(k-a) + \int_{k-m}^{k} F(t)u(t-\rho)dt + \int_k^{k-m+\omega-\rho} E(t)u(t-\omega)dt - \int_k^\infty g(t)dt, \ \mathrm{k} \geq \mathrm{k}_2. \quad (3.2.8)$$

We consider two possible cases.

Case (i):

$u(k)$ is unbounded, i.e., $\limsup_{k\to\infty} u(k) = \infty$.

Thus there exists a sequence of points $\{t_i\}_{i=1}^\infty$ such that $t_i \geq k_2 + \mu_k$, $i = 1,2, \dots . s_i \to \infty$, $u(t_i) \to \infty$ as $i \to \infty$ and $u(t_i) = max\{u(k): k_2 \leq k \leq t_i\}$ $i = 1,2, \dots \dots$

From (3.2.1) and (3.2.8) and the above equality, we get

$$\begin{aligned} u(t_i) &\leq -\eta + G(t_i)u(t_i - a) + \int_{t_i-m}^{t_i} F(t)u(t-\rho)dt + \int_{t_i}^{t_i-m+\omega-\rho} E(t)u(t-\omega)dt - \int_{t_i}^\infty g(t)dt, \\ &\leq -\eta + \left(G(t_i) + \int_{t_i-m}^{t_i} F(t)dt + \int_{t_i}^{t_i-m+\omega-\rho} E(t)dt\right) u(t_i) - \int_{t_i}^\infty g(t)dt \\ &\leq -\eta + u(t_i) - \int_{t_i}^\infty g(t)dt. \end{aligned}$$

Let $i \to \infty$, and one has $\eta \leq 0$,

Which is contradiction.

Hence, $u(t_i) \geq -\eta + u(t_i) - \int_{t_i}^\infty g(t)dt.$

Case (ii):

$u(k)$ is bounded,i.e.,$lim\, sup_{k\to\infty}\ u(k) = S \in [0, \infty)$. Let$\{\bar{t}_i\}_{i=1}^\infty$be a sequence of points such that $\bar{t}_i \to \infty$ and $u(\bar{t}_i) \to S$ as $i \to \infty$. Let λ_i be such that

$u(\lambda_i) = max\{u(t): \bar{t}_i - \mu_k \le t \le \bar{t}_i - \gamma_k\},\ \ \bar{t}_i - \mu_k \le \lambda_i \le \bar{t}_i - \gamma_k, i = 1,2, \dots \dots$

Then $\lambda_i \to \infty$ as $i \to \infty$ and $lim\, sup_{i\to\infty}\ u(\lambda_i) \le S$.

Thus ,by (3.2.1) and (3.2.8),we have

$$u(\bar{t}_i) \le -\eta + G(\bar{t}_i)u(\bar{t}_i - a) - \int_{\bar{t}_i - m}^{\bar{t}_i} F(t)u(t-\rho)dt + \int_{\bar{t}_i}^{\bar{t}_i - m + \omega - \rho} E(t)u(t-\omega)dt - \int_{\bar{t}_i}^{\infty} g(t)dt,$$

$$\le -\eta + u(\lambda_i) - \int_{\bar{t}_i}^{\infty} g(t)dt,$$

Taking the superior limit as $i \to \infty$ on both sides,we obtain

$$S \le -\eta + \lim_{i\to\infty} sup\ u(\lambda_i) \le -\eta + S,$$

So $\eta \le 0$,

which is also a contradiction.

Hence

$$\left[u(k) - G(k)u(k-a) + \int_k^\infty g(t)dt\right]' + E(k)u(k-\omega) - F(k)u(k-\rho) \le 0$$

has a eventually positive solution

(ii) Let $k_1 > k_0$ such that $u(k) < 0$ for $k \ge k_1 - k_0$.

Then, from (3.2.2) and (3.2.5),we have

$$w'(k) \ge -D(k-m+\omega-\rho)u(k-m-\rho) \ge 0,\ \ k \ge k_1 \quad (3.2.9)$$

So,$w(k)$ is non decreasing for $k \ge k_1$.

If $w(k) < 0$ does not hold, then eventually $w(k) > 0$,which implies that there exists a constant $\eta > 0$ and $k_2 \ge k_1$ such that $w(k) \le \eta$ for $k \ge k_2$.

By (3.2.2) we have

$$u(k) \ge -\eta + G(k)u(k-a) + \int_{k-m}^{k} F(t)u(t-\rho)dt + \int_{k}^{k-m+\omega-\rho} E(t)u(t-\omega)dt - \int_k^\infty g(t)dt,\ \mathrm{k} \ge \mathrm{k}_2. \quad (3.2.10)$$

We further consider two possible cases.

Case (i):

$u(k)$ is unbounded, i.e., $\lim\inf_{k\to\infty} u(k) = -\infty$.

Thus there exists a sequence of points $\{t_i\}_{i=1}^{\infty}$ such that $t_i \ge k_2 + \mu_k,\ \ i = 1,2, \dots . t_i \to \infty,$

$u(t_i) \to \infty$ as $i \to \infty$ and $u(t_i) = min\{u(k): k_2 \le k \le t_i\}\ \ i = 1,2, \dots \dots ..$

From (3.2.1) and (3.2.10) and the above equality,we get

$$u(t_i) \ge -\eta + G(t_i)u(t_i - a) + \int_{t_i - m}^{t_i} F(t)u(t-\rho)dt - \int_{t_i}^{t_i - m + \omega - \rho} E(t)u(t-\omega)dt - \int_{t_i}^{\infty} g(t)dt,$$

$$\le\ -\eta + \left(G(t_i) + \int_{t_i - m}^{t_i} F(t)dt + \int_{t_i}^{t_i - m + \omega - \rho} E(t)dt\right) u(t_i) - \int_{t_i}^{\infty} g(t)dt$$

$$\leq -\eta + u(t_i) - \int_{t_i}^{\infty} g(t)dt.$$

Let $i \to \infty$, and one has $\eta \leq 0$,

Which is contradiction.

Hence,

$$\lim \inf_{k\to\infty} u(k) = -\infty.$$

Case(ii):

$u(k)$ is bounded,i.e.,$\lim inf_{k\to\infty} u(k) = S \in (-\infty, 0]$.

Let$\{\bar{t}_i\}_{i=1}^{\infty}$be a sequence of points such that $\bar{t}_i \to \infty$ and $u(\bar{t}_i) \to S$ as $i \to \infty$.

Let λ_i be such that

$u(\lambda_i) = min\{u(t): \bar{t}_i - \mu_k \leq t \leq \bar{t}_i - \gamma_k\}$, $\bar{t}_i - \mu_k \leq \lambda_i \leq \bar{t}_i - \gamma_k, i = 1,2, \dots \dots$

Then $\lambda_i \to \infty$ as $i \to \infty$ and $\lim inf_{i\to\infty} u(\lambda_i) \geq S$.

Thus ,by (3.2.1) and (3.2.10),we have

$$u(\bar{t}_i) \geq \eta + G(\bar{t}_i)u(\bar{t}_i - a) + \int_{\bar{t}_i-m}^{\bar{t}_i} F(t)u(t-\rho)dt + \int_{\bar{t}_i}^{\bar{t}_i-m+\omega-\rho} E(t)u(t-\omega)dt - \int_{\bar{t}_i}^{\infty} g(t)dt,$$

$$\geq \eta + u(\lambda_i) - \int_{\bar{t}_i}^{\infty} g(t)dt,$$

Taking the inferior limit as $i \to \infty$ on both sides, we obtain

$$S \geq \eta + \lim_{i\to\infty} inf\ u(\lambda_i) \geq -\eta + S,$$

So $\eta \leq 0$,

which is a contradiction.

Hence, $\left[u(k) - G(k)u(k-a) + \int_k^{\infty} g(t)dt\right]' + E(k)u(k-\omega) - F(k)u(k-\rho) \geq 0$ has a eventually negative solution.

Hence the proof.

CONCLUSION:

Throughout this work, we discussed some theorems on oscillatory behavior of solutions of differential equations with positive and negative coefficients and then we discussed the solution is oscillate and non oscillate. Finally we establish that, when the solution of the differential equation is eventually positive and eventually negative. In this paper we conclude that whether the solution is eventually positive then it is said to be oscillate ,otherwise it is said to be non oscillate. our results improve the known results in the research field.

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