OSCILLATION OF SECOND ORDER EMDEN-FOWLER DELAY DYNAMIC EQUATIONS

D.POORNIMA[1], S.KAVITHA[2]

[1]Research Scholar, Department of Mathematics

[2]Assistant Professor, Department of Mathematics

Vivekanandha College of Arts and Sciences For Women (Autonomous), Tamilnadu, India.

ABSTRACT

We establish some new oscillation criteria for the second-order Emden-Fowler delay dynamic equations $y^{\Delta\Delta}(t) + q(t)y^{\beta}\big(\tau(t)\big) = 0$ *on a time scale* $\mathbb{T}$; *here* β *is a quotient of odd positive integers with* $q(t)$ *real-valued positive rd-continuous functions defined on* $\mathbb{T}$. *We apply the qualitative behaviour of these equations on time scales. Our results in this paper not only extend the results given in Oscillation of second-order delay dynamic equations, but also the Oscillation of second order Emden-Fowler delay differential equation and the second-order Emden-Fowler delay difference equation.*

KEYWORDS: *Oscillation, Delay dynamic equation*

1. INTRODUCTION

Considered the second-order delay dynamic equations on time scales

$$y^{\Delta\Delta}(t) + q(t)y\big(\tau(t)\big) = 0 \quad \text{for } t\epsilon\mathbb{T} \tag{1.1}$$

and established some sufficient conditions for oscillation of (1.1).

To the best of our knowledge, there are no results regarding the oscillation of the solutions of the following second order nonlinear delay dynamic equations on time scales up to now

$$y^{\Delta\Delta}(t) + q(t)y^{\beta}\big(\tau(t)\big) = 0 \quad \text{for } t\epsilon\mathbb{T} \tag{1.2}$$

Clearly, (1.1) is the special cases of (1.2). To develop the qualitative theory of delay dynamic equations on time scales, in this paper, we consider the second-order nonlinear delay dynamic equations on time scales (1.2).

As we are interested in oscillatory behaviour, we assume throughout this paper that the given time scale $\mathbb{T}$ is unbounded above, i.e., it is a time scale interval of the form $[a, \infty)$ with $a \in \mathbb{T}$.

We assume that β is a quotient of odd positive integer, $q(t)$ is positive, real-valued rd-continuous functions defined on $\mathbb{T}, \tau(t): \mathbb{T} \to \mathbb{T}$ is an rd-continuous function such that $\tau(t) \leq t$ and $\tau(t) \to \infty (t \to \infty)$.

By a solution of (1.2), we mean a nontrivial real-valued function y satisfying (1.2) for $t \geq t_y \geq a$. A solution y of (1.2) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise it is called nonoscillatory. (1.2) is called oscillatory if all solutions are oscillatory. Our attention is restricted to those solutions y of (1.2) which exist on some half line $[t_y, \infty)$ with $\sup\{|y(t)| : t \geq t_0\} > 0$ for any $t_0 \geq t_y$.

We note that if $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = 0, \mu(t) = 0, y^{\Delta}(t) = y(t)$ and (1.2) becomes the second-order Emden-Fowler delay differential equation

$$y(t) + q(t)y^{\beta}(\tau(t)) = 0 \ for\ t \in \mathbb{R}. \tag{1.3}$$

If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$,

$\mu(t) = 1, y^{\Delta}(t) = \Delta y(t) = y(t+1) - y(t)$

and (1.2) becomes the second-order Emden-Fowler delay difference equation

$$\Delta^2 y(t) + q(t)y^{\beta}(\tau(t)) = 0 \ for\ t \in \mathbb{Z} \tag{1.4}$$

In the case of $\beta > 1$, (1.2) is the prototype of a wide class of nonlinear dynamic equations called Emden-Fowler superlinear dynamic equations, and if $0 < \beta < 1$, then (1.2) is the prototype of dynamic equations called Emden-Fowler sublinear dynamic equations.

2. MAIN RESULTS

Result 1

Assume $y(t)$ is an eventually positive solution of

$$y^{\Delta\Delta}(t) + q(t)y^{\beta}(\tau(t)) = 0 \quad \text{for } t\epsilon\mathbb{T}. \tag{1.1}$$

Then, there exists $t_1 \geq a$ such that

$$y^{\Delta}(t) > 0, \quad y^{\Delta\Delta}(t) < 0, \quad \text{for } t \geq t_1. \tag{1.2}$$

Result 2

Assume $\int_a^{\infty} \sigma(t)q(t)\Delta t = \infty$. (2.1)

Then an eventually positive solution $y(t)$ of

$y^{\Delta\Delta}(t) + q(t)y^{\beta}(\tau(t)) = 0$ for $t\epsilon\mathbb{T}$ satisfies eventually

$y(t) \geq ty^{\Delta}(t)$, $\frac{y(t)}{t}$ is non increasing. (2.2)

Result 3

Let z and y are differentiable on time scale $\mathbb{T}$ with $y(t) \neq 0$ for all $t \in \mathbb{T}$.

Then we have,

$$(y^{\beta})^{\Delta}(\frac{y^2}{y^{\beta}})^{\Delta} = (z^{\Delta})^2 - (yy^{\sigma})^{\beta}\left(\left(\frac{z}{y^{\beta}}\right)^{\Delta}\right)^2. \tag{3.1}$$

Theorem 4

Assume $\int_a^\infty \sigma(t)\, q(t)\, \Delta t = \infty$ holds, $\beta \geq 1$. If

$$\lim_{t\to\infty} \sup \{t \int_t^\infty q(s) (\tfrac{\tau(s)}{s})^\beta \Delta s\} = \infty, \quad (4.1)$$

then $y^{\Delta\Delta}(t) + q(t)\, y^\beta(\tau(t)) = 0$ for $t\epsilon\mathbb{T}$ is oscillatory on $[a, \infty)$.

Proof

Suppose that $y^{\Delta\Delta}(t) + q(t)\, y^\beta(\tau(t)) = 0$ for $t\epsilon\mathbb{T}$ has a nonoscillatory solution $y(t)$.

We may assume that $y(t) > 0$ and $y(\tau(t)) > 0$ for all $t \geq t_1 > a$.

We shall consider only this case, since the proof when $y(t)$ is eventually negative is similar.

By using result (2) we get (1.2) holds. From (1.1), (1.2) we have for $T \geq t \geq t_1$.

$$\int_t^T q(s) y^\beta(\tau(s))\Delta s = -\int_t^T y^{\Delta\Delta}(s)\Delta s = y^\Delta(t) - y^\Delta(T) \leq y^\Delta(t),$$

and hence $\int_t^\infty q(s)\, y^\beta(\tau(s))\Delta s \leq y^\Delta(t)$.

This and result (2) provide for sufficiently large $t \in \mathbb{T}$

$$y(t) \geq t y^\Delta(t) \geq t \int_t^\infty q(s)\, y^\beta(\tau(s))\Delta s$$

$$\geq t \int_t^\infty q(s) \left(\frac{\tau(s)}{s}\right)^\beta y^\beta(s)\, \Delta s$$

$$\geq y^\beta(t)\{t \int_t^\infty q(s) \left(\frac{\tau(s)}{s}\right)^\beta \Delta s\}.$$

So, $t \int_t^\infty q(s) (\frac{\tau(s)}{s})^\beta \Delta s \leq (\frac{1}{y(t)})^{\beta-1}$.

Here $\beta \geq 1$ and (1.2) imply

$$t \int_t^\infty q(s) \left(\frac{\tau(s)}{s}\right)^\beta \Delta s \leq \left(\frac{1}{y(t_1)}\right)^{\beta-1}.$$

Which is a contradiction.

Hence the proof.

Theorem 5

Assume $\int_a^\infty \sigma(t)\, q(t)\, \Delta t = \infty$ holds, $\beta \geq 1$.

Furthermore, assume that there exists a function $z(t) \in c_{rd}^1([a,\infty),\mathbb{R})$ such that

$$\lim_{t\to\infty} sup \int_a^t (q(s)(\frac{\tau(s)}{\sigma(s)})^\beta \left(z(\sigma(s))\right)^2 - M^{\beta-1}\left(z^\Delta(s)\right)^2\Big) = \infty \qquad (5.1)$$

holds for all constants $M > 0$.

Then $yy^{\Delta\Delta}(t) + q(t)y^\beta(\tau(t)) = 0$ for $t\epsilon\mathbb{T}$ is oscillatory on $[a,\infty)$.

Proof

Suppose that $y^{\Delta\Delta}(t) + q(t)y^\beta(\tau(t)) = 0$ for $t\epsilon\mathbb{T}$ has a nonoscillatory solution $y(t)$.

We may assume without loss of generality that $y(t) > 0$ and $y(\tau(t)) > 0$ for all $t \geq t_1 > a$.

So by result (1.1), (1.2) holds.

Define the function $\omega(t)$ by

$$\omega = \frac{z^2 y^\Delta}{y^\beta}. \qquad (5.2)$$

Then, and using the following equation

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t)$$

$$= f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)).$$

We get,

$$\omega^\Delta = (\frac{z^2}{y^\beta})^\sigma y^{\Delta\Delta} + y^\Delta\left(\frac{z^2}{y^\beta}\right)^\Delta.$$

So, from (1.2) and result (3) we have

$$\omega^\Delta = -q(z^\sigma)^2\left(\frac{y\circ\tau}{y^\sigma}\right)^\beta + \frac{y^\Delta}{(y^\beta)^\Delta}(z^\Delta)^2 - \frac{y^\Delta}{(y^\beta)^\Delta}(yy^\sigma)^\beta\left(\left(\frac{z}{y^\beta}\right)^\Delta\right)^2. \qquad (5.3)$$

By using result (2),

$$\frac{y(\tau(t))}{\tau(t)} \geq \frac{y(\tau(t))}{\sigma(t)},$$

Here $\beta \geq 1$,

$$((y(t))^\beta)^\Delta = \beta\int_0^1 [hy^\sigma + (1-h)y]^{\beta-1}y^\Delta(t)\,dh,$$

and $y^\Delta(t) > 0, \quad y^{\Delta\Delta}(t) < 0, \quad for\ t \geq t_1.$

Imply $(y^\beta)^\Delta(t) \geq \beta\left(y(t)\right)^{\beta-1}y^\Delta(t) \geq \beta\left(y(t_1)\right)^{\beta-1}y^\Delta(t),$

then $\omega^\Delta \leq -q(\frac{\tau}{\sigma})^\beta + M^{\beta-1}(z^\Delta)^2,$

where $M = (\beta^{\frac{1}{\beta-1}} y(t_1))^{-1}$, if $\beta > 1$. If $\beta = 1$,

we choose $M = 1$.

Therefore,

$$\int_{t_1}^{t} (q(s)\left(\frac{\tau(s)}{q(s)}\right)^{\beta} (z(\sigma(s)))^2 - M^{\beta-1}(z^{\Delta}(s))^2)\Delta s \leq -\int_{t_1}^{t} \omega^{\Delta}(s)\Delta s \leq \omega(t_1),$$

Which contradicts (4.1).

Hence the proof.

Theorem 6

Assume $\int_a^{\infty} \sigma(t)\, q(t)\, \Delta t = \infty$ holds, $\beta > 1$. Furthermore,

$$\lim_{t\to\infty} sup \int_a^t q(s)\,\sigma(s)\left(\frac{\tau(s)}{\sigma(s)}\right)^{\beta} \Delta s = \infty. \quad (6.1)$$

Then $y^{\Delta\Delta}(t) + q(t)\, y^{\beta}(\tau(t)) = 0$ for $t\epsilon\mathbb{T}$ is oscillatory on $[a, \infty)$.

Proof

We assume that $y^{\Delta\Delta}(t) + q(t)\, y^{\beta}(\tau(t)) = 0$ has a nonoscillatory solution such that

$y(t) > 0$ and $y(\tau(t)) > 0$ for all $t \geq t_1 > a$.

By result (1.1) we obtain

$y^{\Delta}(t) > 0, \quad y^{\Delta\Delta}(t) < 0, \quad for\ t \geq t_1.$

Now we let $z = \sqrt{1}$ and define the Riccati substitution ω by (5.2).

Using the product rule from following equation

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t)$$

$$= f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).$$

and result (1.2), we get,

$$\omega^{\Delta} = \{y^{\Delta} + \sigma(t)\, y^{\Delta\Delta}\}(y^{-\beta})^{\sigma} + t y^{\Delta}(y^{-\beta})^{\Delta}$$

$$= y^{\Delta}(y^{-\beta})^{\sigma} - \sigma(t)\, q(t)\left(\frac{y(\tau(t))}{y(\sigma(t))}\right)^{\beta} + (t y^{\Delta})(y^{-\beta})^{\Delta}$$

$$\leq \frac{(y^{\beta-1})^{\Delta}}{1-\beta} - \sigma(t)\, q(t)\left(\frac{\tau(t)}{\sigma(t)}\right)^{\beta},$$

where the last inequality is true because $(y^{-\beta})^{\Delta} \leq 0$ due to

$$((y(t))^{\beta})^{\Delta} = \gamma \int_0^1 [hy^{\sigma} + (1-h)y]^{\gamma-1} y^{\Delta}(t)\, dh,$$

and

$$y^{\Delta}(t) > 0, \qquad y^{\Delta\Delta}(t) < 0, \quad for\ t \geq t_1.$$

because

$$((y(t))^{1-\beta})^{\Delta} = (1-\beta) \int_0^1 [hy^{\sigma} + (1-h)y]^{-\beta} y^{\Delta}(t)\, dh$$

$$\leq (1-\beta) \int_0^1 [hy^{\sigma} + (1-h)y^{\sigma}]^{-\beta} y^{\Delta}(t)\, dh$$

$$= (1-\beta)\left(y^{\sigma}(t)\right)^{-\beta} y^{\Delta}(t).$$

Integrating we get,

$$\int_{t_1}^{t} \sigma(t) q(s) \left(\frac{\tau(s)}{\sigma(s)}\right)^{\beta} \Delta s \leq \int_{t_1}^{t} \left\{\frac{y^{1-\beta}}{1-\beta} - \omega\right\}^{\Delta} (s) \Delta s$$

$$= \frac{y^{1-\beta}(t)}{1-\beta} - \omega(t) - \frac{y^{1-\beta}(t_1)}{1-\beta} + \omega(t_1)$$

$$\leq \frac{y^{1-\beta}(t_1)}{1-\beta} + \omega(t_1).$$

Which is contradiction to $\lim_{t\to\infty} sup \int_a^t q(s)\, \sigma(s) \left(\frac{\tau(s)}{\sigma(s)}\right)^{\beta} \Delta s = \infty$

Hence the proof.

CONCLUSIONS

Throughout this work, we discussed some results and theorems on oscillation of second order Emden-foweler delay dynamic equations and then we discussed a second-order nonlinear delay dynamic equations on time scales. We establish some new oscillation criteria for the second-order Emden-Fowler delay dynamic equations. So the solution of the second order non-linear equation is oscillatory.

REFERENCES

[1] R.P. Agarwal, M.Bohner, D. O'Regan, A. Peterson, Dynamic equations on time scales: A survey, J. Comput. Appl. Math. 141 (1–2) (2002) 1–26.
[2] R.P. Agarwal, D. O'Regan, S.H. Saker, Oscillation criteria for second-order nonlinear neutral delay dynamic equations,J. Math. Anal. Appl. 300 (2004) 203–217.
[3] R.P. Agarwal, S.L. Shieh, C.C. Yeh, Oscillation criteria for second order retarded differential equations, Math. Comput. Modelling 26 (4) (1997) 1–11.
[4] S. Chen, L.H. Erbe, Riccati techniques and discrete oscillations, J. Math. Anal. Appl. 142 (1989) 468–487.