

# On the Linear Difference equations

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## ABSTRACT;

In this paper, I have solved some problems by using First order homogeneous equation and Second order homogeneous equation

**Keywords;** First order homogeneous, Second order homogeneous equation, Linearity principle, Fibonacci sequence, flipping coins

## INTRODUCTION

We have just seen that some First order homogeneous, Second order homogeneous equation can be solved using the some important methods in this paper

### First order homogeneous equation:

Think of the time being discrete and taking integer values  $n = 0, 1, 2,$  and  $x(n)$  describing the state of some system at time  $n$ . We consider an equation of the form

$$\text{First order homogeneous } x(n) = ax(n - 1)$$

where  $x(n)$  is to be determined is a constant. This equation is called a *first order homogeneous* equation and it is easy to solve iteratively.

$$x(n) = ax(n - 1) = a(ax(n - 2)) = a^2x(n - 2) = \dots = a^n x(0).$$

So if we are given  $x(0)$ , i.e. the state of the system at time 0, then the state of the system at time  $n$  is given by  $x(n) = a^n x(0)$ , i.e. this is a model for exponential growth or decay.

To summarize

The general solution of  $x(n) = ax(n - 1)$  is  $x(n) = Ca^n$

### Interest rate:

A bank account has a yearly interest rate of 5% compounded monthly. If you invest \$1000, how much money do you have after 5 years? Since the interest is paid monthly we set

$$x(n) = \text{amount of money after } n \text{ months and since}$$

we get one twelfth of 5% every month we have

$$x(n) = \left(1 + \frac{.05}{12}\right) x(n-1) = \left(1 + \frac{1}{240}\right) x(n-1) = \left(\frac{241}{240}\right) x(n-1)$$

and so after 5 year we have with  $x(0) = 1000$

$$x(60) = \left(\frac{241}{240}\right)^{60} 1000 = 1283.35$$

### First order inhomogeneous equation:

Let us consider an equation of the form

$$\text{First order inhomogeneous } x(n) = ax(n-1) + b(n)$$

where  $b(n)$  is a given sequence and  $x(n)$  is unknown. For example we may take

$$b(n) = b, \quad b(n) = 2n^2 + 3, \quad b(n) = b3^n.$$

This equation is called *inhomogeneous* because of the term  $b(n)$ . The following simple fact is useful to solve such equations

### Linearity principle:

Suppose  $x(n)$  is a solution of the homogeneous first order equation  $x(n) = ax(n-1)$  and  $y(n)$  is a solution of the inhomogeneous first order equation  $y(n) = ay(n-1) + b(n)$ .

Then  $z(n) = x(n) + y(n)$  is a solution of the inhomogeneous equation  $z(n) = az(n-1) + b(n)$ . Indeed we have

$$\begin{aligned} z(n) &= x(n) + y(n) \\ &= ax(n-1) + ay(n-1) + b(n) \\ &= a[x(n-1) + y(n-1)] + b(n) \\ &= az(n-1) + b(n). \end{aligned}$$

To find the general solution of a first order homogeneous equation we need

- Find *one particular* solution of the inhomogeneous equation.
- Find the general solution of the homogeneous equation. This solution has a free constant in it which we then determine using for example the value of  $x(0)$ .
- The general solution of the inhomogeneous equation is the sum of the particular solution of the inhomogeneous equation and general solution of the homogeneous equation.

**Example:** Solve

$$x(n) = ax(n-1) + b$$

i.e., the inhomogeneous term is  $b(n) = b$  is constant. We look for a particular solution, and after some head scratching we try  $x(n) = D$  to be constant and find

$$D = aD + b,$$

$$\text{or} \quad \frac{b}{1 - a}$$

$$D =$$

The general solution is then

$$x(n) = Ca^n + \frac{b}{1 - a}.$$

**Example:** Solve

$$2x(n) - x(n - 1) = 2^n, \quad x(0) = 3$$

The solution of the homogenous equation  $2x(n) - x(n - 1)$  is  $x(n) = C(1/2)^n$ . To find a particular solution of the inhomogeneous problem we try an exponential function  $x(n) = D2^n$  with a constant  $D$  to be determined. Plugging into the equation we find

$$2D2^n - D2^{n-1} = 2^n$$

or after dividing by  $2^{n-1}$

$$4D - D = 2 \text{ or } D = \frac{2}{3}.$$

So the general solution is

$$x(n) = C \left(\frac{1}{2}\right)^n + \frac{2}{3}2^n.$$

and the initial condition gives  $x(0) = 3 = C + \frac{2}{3}$  and so

$$x(n) = \frac{7}{3} \left(\frac{1}{2}\right)^n + \frac{2}{3}2^n.$$



**More interest rate:**

A bank account gives an interest rate of 5% compounded monthly. If you invest initially \$1000, and add \$10 every month. How much money do you have after 5 years? Since the interest is paid monthly we set

$x(n)$  = amount of money after  $n$  months and we have

the equation for  $x(n)$

$$x(n) = \left(1 + \frac{.05}{12}\right) x(n-1) + 10 = \left(\frac{241}{240}\right) x(n-1) + 10$$

For the particular solution we try  $x(n) = D$  and find

$$D = \frac{241}{240}D + 10$$

i.e.,  $D = -2400$ . The general solution is then

$$x(n) = D \left(\frac{241}{240}\right)^n - 2400$$

and  $x(0) = 1000$  gives

$$x(n) = 3400 \left(\frac{241}{240}\right)^n - 2400$$

and so  $x(60) = 1963.41$

**Second order homogeneous equation:**

We consider an equation where  $x(n)$  depends on both  $x(n-1)$  and  $x(n-2)$ :

$$\text{Second order homogeneous } x(n) = ax(n-1) + bx(n-2).$$

It is easy to see that we are given both  $x(0)$  and  $x(1)$  we can then determine  $x(2)$ ,  $x(3)$ , and so on.

**Linearity Principle:**

*One verifies verify that if  $x(n)$  and  $y(n)$  are two solutions of the second order homogeneous equation, then  $C_1x(n)+C_2y(n)$  is also a solution for any choice of constants  $C_1, C_2$ .*

To find the general solution we get inspired by the homogeneous first order equation and look for solutions of

the form

$$x(n) = \alpha^n$$

If we plug this into the equation we find

$$a^n = a\alpha^{n-1} + b\alpha^{n-2}$$

and dividing by  $\alpha^{n-2}$  give

$$\alpha^2 - a\alpha + b = 0$$

We find (in general) two distinct roots  $\alpha_1$  and  $\alpha_2$  and the general solution has then the form

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$$\text{General solution } x(n) = C_1\alpha_1^n + C_2\alpha_2^n$$


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**Example:**

The **Fibonacci sequence** is given by

$$x(n) = x(n-1) + x(n-2), \quad x(0) = 0, x(1) = 1$$

that is every term of the sequence is the sum of the two preceding terms. It is given by 0, 1, 1, 2, 3, 5, 8, 13, 21,

34, 55, 89, 144, 233 ···

As we will see, the golden ratio

$$\varphi = \frac{1 + \sqrt{5}}{2} = 1.61803398875$$

occurs in the Fibonacci sequence in the sense that for large  $n$

$$\frac{x(n+1)}{x(n)} \approx \varphi.$$

For example  $89/55 = 1.61818181818$ ,  $144/89 = 1.61797752809$ ,  $233/144 = 1.61805555556$ , and so on... To see why it occurs we solve the second order difference equation: with  $x(n) = \alpha^n$  we find

$$\alpha^2 - \alpha - 1 = 0$$

or

$$\alpha = \frac{1 \pm \sqrt{5}}{2}$$

So the the general solution is

$$x(n) = C_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + C_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n.$$

and with  $x(0) = 0$  and  $x(1) = 1$  we find

$$x(n) = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right].$$

Since  $\left| \frac{1 - \sqrt{5}}{2} \right| < 1$  the second term is vanishingly small for large  $n$  so  $x(n) \approx \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n$ .

### Example:

The **Fibonacci sequence and flipping coins.**

The Fibonacci sequence shows up in many instances. In a probabilistic context it shows up in the

Determine the probability to flip a coin  $n$  times and have no successive heads.

following problem:

To do this we need to *count* the number of sequences of heads (H) and tails (T) such that no successive heads occurs.

So we set

$$f(n) = \text{number of sequences of } n \text{ H or T without consecutive H}$$

and then we have

$$P\{\text{flip a coin } n \text{ times without consecutive heads}\} = \frac{f(n)}{2^n}$$

To find  $f(n)$  we derive a recursive relation for it. Suppose we have a sequence of length  $n$  which ends up with a  $T$ . Then we can put in the first  $n - 1$  spots any sequence with no consecutive heads and this creates a sequence of length  $n$  without consecutive heads. There are  $f(n - 1)$  such sequences. If the sequence of length  $n$  ends up with a  $H$  then the  $n - 1^{\text{th}}$  entry in the sequence needs to be  $T$ , one obtains then a sequence without consecutive heads if the first  $n - 2$  entries any sequence without consecutive heads. There are  $f(n - 2)$

such sequences and thus we found that

$$f(n) = f(n - 1) + f(n - 2).$$

If  $n = 1$  then we have  $f(1) = 2$  and if  $n = 2$  we have  $f(2) = 3$  so that we obtain the Fibonacci sequence gain but shifted by two:

$$f(n) = x(n + 2) = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+2} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+2} \right]$$

As an example we find that the probability to flip a coin 15 times and have no successive heads is

$$\frac{x(17)}{2^{15}} = 0.0487.$$

**Second order inhomogeneous equation:**

We consider an equation of the form

Second order homogeneous  $x(n) = ax(n - 1) + bx(n - 2) + c(n).$

where  $x(n)$  is unknown and  $c(n)$  is a fixed sequence. As for first order equations we can solve such equations by

1. Solve the homogeneous equation  $x(n) = ax(n - 1) + bx(n - 2).$
2. Find a particular solution of the inhomogeneous equation.
3. Write the general solution as the sum of the particular inhomogeneous equation plus the general solution of the homogeneous equation.

**Example:** Find the general solution of the second order equation  $3x(n) + 5x(n - 1) - 2x(n - 2) = 5.$  For the homogeneous equation  $3x(n) + 5x(n - 1) - 2x(n - 2) = 0$  let us try  $x(n) = \alpha^n$  we obtain the quadratic equation

$$3\alpha^2 + 5\alpha - 2 = 0 \text{ or } \alpha = 1/3, -2 \text{ and so the}$$

general solution of the homogeneous equation is

$$x(n) = C_1 \left(\frac{1}{3}\right)^n + C_2(-2)^n$$

For a particular equation  $3x(n) + 5x(n-1) - 2x(n-2) = 5$  we try  $x(n) = D$  and find

$$3D + 5D - 2D = 5$$

i.e.  $D = 5/6$  and so the general solution is

$$x(n) = \frac{5}{6} + C_1 \left(\frac{1}{3}\right)^n + C_2(-2)^n$$

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