One Numerical Simulation for the 2D Variable Order Fractional Schrodinger Equation with the Quantum Riesz-Feller Derivative

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Abstract

In this paper, we present an accurate numerical method for solving a space-fractional Schrodinger equation in two dimensions. The quantum Riesz–Feller fractional derivative is used to define the fractional derivatives. The proposed numerical method is analogue of the Crank–Nicholson method and right-shifted formula of Grunwald-Letnikov

Keys Words : *fractional derivatives- quantum Riesz–Feller derivative-2D Variable Order*

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1. Introduction

It is well known that one of the most important partial differential equations in mathematical physics is the Schrodinger equation that describes the change of the quantum behavior of some physical systems with time. This equation was formulated in 1925, and published in 1926, by the Austrian physicist Erwin Schrodinger.

In a sequence of papers ([8], [9], [10]) Nick Laskin constructed the fundamental equation of fractional quantum mechanics i.e., Time Dependent Fractional Schrodinger Equation (TFSE), in the following form:

$$i\frac{h}{2\pi}\frac{\partial\Psi(r,t)}{\partial t}=C_{\alpha}(m)(-\Delta)^{\frac{\alpha}{2}}\Psi(r,t)+V(r,t)\Psi(r,t)\quad t\geq 0, r\in\mathbb{R}\quad (1)$$

for the wave function Ψ of a quantum particle with the mass m that moves in a potential field with the potential *V*. Where h is the Plank constant, $C_{\alpha}(m)$ is a positive constant which equals $\frac{h^2}{2m}$ for $\alpha=2$ and $(-\Delta)^{\frac{\alpha}{2}}$ was called the quantum Riesz fractional derivative of order α. In the mathematical literature, $(-\Delta^{\frac{\alpha}{2}})$ is usually referred to as the fractional Laplacian. For $\alpha=2$, the quantum Riesz fractional derivative becomes the negative Laplace operator $-\Delta$ and eq (1) is reduced to the classical Schrodinger equation for a quantum particle with the mass m that moves in a potential field with the potential V

The main goal of this paper is to develop an accurate numerical algorithm for approximating the numerical solutions of the variable order fractional Schrodinger equation with the quantum Riesz–Feller derivative for a particle that moves in a 2-D potential field, using the right schifted formula Grunwald-Letnikov[1] [2],

Literature review reveals many publications ([5], [6], [7]-[10], [11]), which introduce several analytical and numerical methods for one or multidimensional space-fractional and space-time-fractional Schrodinger equations with some specific potential fields including the zero potential (free particle), the potential, the infinite potential well, the Coulomb potential, and the rectangular barrier.

This paper is structured as follows: in the next section we introduce some definitions on fractional calculus and some properties of non-standard discretization. Section 3 is devoted to discretization.. In Section 4, some numerical treatments are established with their results.

Concluding are given in Section 5.

2 Preliminaries and notations

In the following we give some preliminary results which are needed in subsequent sections of this paper.

2.1 Fractional calculus definitions

In the labels, many different definitions of the fractional derivatives were introduced (see [10, 14, 15, 16]). The time-fractional derivatives are often given in the Caputo,Riemann-Liouville, or Grunwald-Letnikov sense. As to the space-fractional derivative, it is usually defined as an operator inverse to the Riesz potential and is referred to as the Riesz fractional derivative. Podlubny concludes ([14]) that "the complete theory of fractional differential equations,especially the theory of boundary value problems for fractional differential equations,can be developed only with the use of both left and right derivatives. So the spatial derivatives discussed in this paper are the fractional Riesz-Feller potential operator, which includes the left and right Riemann-Liouville fractional derivatives.

For $0 < \alpha < 2$, $\alpha \neq 1$ and $|\theta| \leq \{\alpha, 2-\alpha\}$, the quantum Riesz-Feller fractional derivative D_θ^α was represented in the following form (see [15, 16]):

Definition 2.1.

$$D_\theta^\alpha u(x) = (c_+ D_+^\alpha + c_- D_-^\alpha) u(x) \tag{2.1}$$

where the coefficients $c\pm$ are given by

$$c_+ = c_+(\alpha,\theta) = \frac{\sin((\alpha-\theta)\frac{\pi}{2})}{\sin(\alpha\pi)} \qquad c_- = c_-(\alpha,\theta) = \frac{\sin((\alpha+\theta)\frac{\pi}{2})}{\sin(\alpha\pi)} \tag{2.2}$$

$$(D_+^\alpha u)(x) = (\frac{d}{dx})^n (I_+^{n-\alpha} f)(x), \qquad (D_-^\alpha u)(x) = (-\frac{d}{dx})^n (I_-^{n-\alpha} u)(x) \tag{2.3}$$

are the left-side and right-side Riemann-Liouville fractional derivatives with $x \in \mathbb{R}$ and $\alpha > 0, n-1 < \alpha \leq n, n = 1,2$ In expressions (2.3) the fractional operators $I_\pm^{n-\alpha}$ are defined as the left- and right-side of Weyl fractional integrals, which given by

$$(I_+^\alpha u)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{u(\xi)}{(x-\xi)^{1-\alpha}}, \qquad (I_-^\alpha u)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \frac{u(\xi)}{(x-\xi)^{1-\alpha}} \tag{2.4}$$

For $\alpha = 1$, the representation (2.1) is not valid and has to be replaced by the formula

$$D_0^1 u(x) = \left[\cos(\theta\frac{\pi}{2}) D_0^1 - \sin(\theta\frac{\pi}{2}) D \right] u(x) \tag{2.5}$$

where D refers for the first standard derivative and the operator D_0^1 is related to the Hilbert transform as first noted by Feller in 1952 in his pioneering paper [17]

$$D_0^1 u(x) = \frac{1}{\pi} \frac{d}{dx} \int_{-\infty}^{+\infty} \frac{u(\xi)}{x-\xi} d\xi$$

We can naturally expand this definition to a variable-order quantum Riesz-Feller fractional derivative $D_{\theta(x)}^{\alpha(x)}$ for $0 < \alpha(x) < 2, \alpha(x) \neq 1$ and $|\theta(x) \leq \min\{\alpha(x), 2-\alpha(x)\}|$ as following

Definition 2.2 For

$$D_{\theta(x)}^{\alpha(x)} u(x) = (c_+ D_+^{\alpha(x)} + c_- D_-^{\alpha(x)}) u(x) \tag{2.6}$$

$$c_+ = c_+(\alpha(x),\theta(x)) = \frac{\sin((\alpha(x)-\theta(x))\frac{\pi}{2})}{\sin(\alpha(x)\pi)}, \quad c_- = c_-(\alpha(x),\theta(x)) = \frac{\sin((\alpha(x)+\theta(x))\frac{\pi}{2})}{\sin(\alpha(x)\pi)} \tag{2.7}$$

2.2 Right-shifted formula of Grunwald-Letnikov

The shifted Grünwald formula for discretizing the two-sided fractional derivative is proposed by Meerschaert and Tadjeran [18], and it was shown that the standard (i.e., unshifted) Grünwald formula to discretize the fractional diffusion equation results in an unstable finite difference scheme regardless of whether the resulting finite difference method is an explicit or an implicit system. Hence, we discretise the Riesz-Feller fractional derivative $D_\theta^\alpha u$ by the shifted Grünwald formula [18]:

If the spatial domain is $[0,L]$ The mesh is N equals intervals of $\delta=\frac{L}{N}$ and $x_l = l\delta$ for $0\le l\le N$, and $u(l\delta)=u_l$

$$ {}_0D_x^\alpha u(x_l)=\frac{1}{\delta^\alpha}\sum_{j=0}^{l+1} g_j u_{l-j+1}+O(\delta) \qquad {}_xD_L^\alpha u(x_l)=\frac{1}{\delta^\alpha}\sum_{j=0}^{N-l+1} g_j u_{l+j-1}+O(\delta) \text{ [1]} \quad (2.8)$$

Where the coefficients are defined by

$$g_0=1 \qquad g_j=(-1)^j\frac{\alpha(\alpha-1).........(\alpha-j+1)}{j!} \quad (2.9)$$

We also have $g_j=(1-\frac{\alpha+1}{j})g_{j-1}$ $\quad j=1,........,N$ [1] (2.10)

In this paper, we consider the fractional Schrodinger equation with the quantum Riesz–Feller derivative that describes the wave function Ψ of a quantum particle that moves in a potential field in two dimensional space with the potential V in the form:

$$ih\frac{\partial\Psi(x,y,t)}{\partial t}=C_{\alpha(x,,t)}(m)(D^{\alpha(x,y,t)}_{\theta(x,y,t)\,|x}+D^{\alpha(x,y,t)}_{\theta(x,y,t)\,|y})\Psi(x,y,t)+V(x,y,t)\Psi(x,y,t) \quad (2.11)$$

$$t>0\ ;\ (x,y)\in[0,L_x]\times[0,L_y] \qquad \Psi(x,y,0)=f(x,y)$$

where $(D^{\alpha(x,y,t)}_{\theta(x,y,t)\,|x}+D^{\alpha(x,y,t)}_{\theta(x,y,t)\,|y})\Psi(x,y,t)=D^{\alpha(x,y,t)}_{\theta(x,y,t)\,|x}\Psi(x,y,t)+D^{\alpha(x,y,t)}_{\theta(x,y,t)\,|y}\Psi(x,y,t)$

In this case $g_0=1 \quad g_j(x)=(-1)^j\frac{\alpha(x)(\alpha(x)-1).........(\alpha(x)-j+1)}{j!}$

3 Discretization

Assume that the coordinates of the mesh points are :

If $t\in[0,T]$; $x\in[0,L_x]$; $y\in[0,L_y]$

$$x_n=nh_x \quad n=0,1,.........,N_x\ ; \qquad y_p=ph_y \quad p=0,1,.........,N_y$$

$$t_k=k\Delta \quad k=0,1,.......,K$$

where $h_x=x_n-x_{n-1} \qquad h_y=y_p-y_{p-1} \qquad \Delta=t_m-t_{m-1}$

We define the approximation of the function $\Psi(x,y,t)$ on the grid (x_n,y_p,t_k) by

$$\Psi(x_n,y_p,t_k)=\Psi^k_{n,p}$$

$$\cdot D^{\alpha(x,y,t)}_{\theta(x,y,t)|x}\Psi(x_n,y_p,t_k)=D^{\alpha(x_n,y_p,t_k)}_{\theta(x_n,y_p,t_k)|x}\Psi^k_{n,p}=(c^k_{+,n,p}\,{}_0D_x^{\alpha^k_{n,p}}+c^k_{-,n,p}\,{}_xD_{L_x}^{\alpha^k_{n,p}})(\Psi^k_{n,p})=$$

$$\frac{1}{(h_x)^{\alpha^k_{n,p}}}[c^k_{+,n,p}\sum_{j=0}^{n+1}g^k_{j,n,p}\Psi^k_{n-j+1,p}+c^k_{-,n,p}\sum_{j=0}^{N_x-n+1}g^k_{j,n,p}\Psi^k_{n+j-1,p}]+O(h_x) \quad (3.1)$$

with $\cdot c^k_{+,n,p}=c_+(x_n,y_p,t_k) \qquad c^k_{-,n,p}=c_-(x_n,y_p,t_k) \qquad \alpha^k_{n,p}=\alpha(x_n,y_p,t_k)$

$\theta^m_{n,p}=\theta(x_n,y_p,t_m)$

$$g^k_{j,n,p}=g_j(x_n,y_p,t_k)=(-1)^j\frac{\alpha(x_n,y_p,t_k)(\alpha(x_n,y_p,t_k)-1).........(\alpha(x_n,y_p,t_k)-j+1)}{j!} \quad (3.2)$$

Applying the usual approximation to a derivative on time and interpolation designs with extrapolation, as a result we receive analog of Crank-Nicholson method in (2.11) [2]

$$ih\frac{\partial\Psi(x_n,y_p,t_k)}{\partial t}=C_{\alpha(x_n,y_p t_k)}(m)(D^{\alpha(x_n,y_p,t_k)}_{\theta(x,y_p,t_k)|x}+D^{\alpha(x_n,y_p,t_k)}_{\theta(x_n,y_p,t_k)|y})\Psi(x_n,y_p,t_k)+V(x_n,y_p,t_k)\Psi(x_n,y_p,t_k)$$

(3.3)

$$ih\frac{\Psi^{k+1}_{n,p}-\Psi^{k}_{n,p}}{\Delta}=\frac{C_{\alpha^{k+1}_{n,p}}(m)(D^{\alpha^{k+1}_{n,p}}_{\theta^{k+1}_{n,p}|x}+D^{\alpha^{k+1}_{n,p}}_{\theta^{k+1}_{n,p}|y})\Psi^{k+1}_{n,p}+C_{\alpha^{k}_{n,p}}(m)(D^{\alpha^{k}_{n,p}}_{\theta^{k}_{n,p}|x}+D^{\alpha^{k}_{n,p}}_{\theta^{k}_{n,p}|y})\Psi^{k}_{n,p}}{2}+V^{k}_{n,p}\Psi^{k}_{n,p} \quad (3.4)$$

where $V^k_{n,p}=V(x_n,y_p,t_k)$ .it is an extension of the expression in 1D . see [19]

$$\square ih\frac{\Psi^{k+1}_{n,p}-\Psi^{k}_{n,p}}{\Delta}=\frac{C_{\alpha^{k+1}_{n,p}}(m)(D^{\alpha^{k+1}_{n,p}}_{\theta^{k+1}_{n,p}|x}+D^{\alpha^{k+1}_{n,p}}_{\theta^{k+1}_{n,p}|y})\Psi^{k+1}_{n,p}+C_{\alpha^{k}_{n,p}}(m)(D^{\alpha^{k}_{n,p}}_{\theta^{k}_{n,p}|x}+D^{\alpha^{k}_{n,p}}_{\theta^{k}_{n,p}|y})\Psi^{k}_{n,p}}{2}+V^{k}_{n,p}\Psi^{k}_{n,p}$$

$$\square 2\Psi^{k+1}_{n,p}+i\frac{1}{h}\Delta C_{\alpha^{k+1}_{n,p}}(m)(D^{\alpha^{k+1}_{n,p}}_{\theta^{k+1}_{n,p}|x}+D^{\alpha^{k+1}_{n,p}}_{\theta^{k+1}_{n,p}|y})\Psi^{k+1}_{n,p}=(2-2i\Delta\frac{1}{h}V^{k}_{n,p})\Psi^{k}_{n,p}-i\frac{1}{h}\Delta C_{\alpha^{k}_{n,p}}(m)(D^{\alpha^{k}_{n,p}}_{\theta^{k}_{n,p}|x}+D^{\alpha^{k}_{n,p}}_{\theta^{k}_{n,p}|y})\Psi^{k}_{n,p}$$

$$\square 2\Psi^{k+1}_{n,p}+i\frac{\Delta C_{\alpha^{k+1}_{n,p}}}{h}[\frac{1}{h_x^{\alpha^{k+1}_{n,p}}}[(c^{k+1}_{+,n,p}\sum_{j=0}^{n+1}g^{k+1}_{j,n,p}\Psi^{k+1}_{n-j+1,p}+c^{k+1}_{-,n,p}\sum_{j=0}^{N_x-n+1}g^{k+1}_{j,n,p}\Psi^{k+1}_{n+j-1,p}]+$$

$$\frac{1}{h_y^{\alpha^{k+1}_{n,p}}}[(c^{k+1}_{+,n,p}\sum_{j=0}^{p+1}g^{k+1}_{j,n,p}\Psi^{k+1}_{n,p-j+1}+c^{k+1}_{-,n,p}\sum_{j=0}^{N_y-p+1}g^{k+1}_{j,n,p}\Psi^{k+1}_{n,p+j-1}]]=(2-2i\Delta\frac{1}{h}V^{k}_{n,p})\Psi^{k}_{n,p}$$

$$-i\frac{\Delta C_{\alpha^{k}_{n,p}}}{h}[\frac{1}{h_x^{\alpha^{k}_{n,p}}}[(c^{k}_{+,n,p}\sum_{j=0}^{n+1}g^{k}_{j,n,p}\Psi^{k}_{n-j+1,p}+c^{k}_{-,n,p}\sum_{j=0}^{N_x-n+1}g^{k}_{j,n,p}\Psi^{k}_{n+j-1,p}]+$$

$$\frac{1}{h_y^{\alpha^{k}_{n,p}}}[(c^{k}_{+,n,p}\sum_{j=0}^{p+1}g^{k}_{j,n,p}\Psi^{k}_{n,p-j+1}+c^{k}_{-,n,p}\sum_{j=0}^{N_y-p+1}g^{k}_{j,n,p}\Psi^{k}_{n,p+j-1}]]$$

$$\square 2\Psi^{k+1}_{n,p}+i\frac{\Delta C_{\alpha^{k+1}_{n,p}}}{h}[\frac{1}{h_x^{\alpha^{k+1}_{n,p}}}[(c^{k+1}_{+,n,p}(g^{k+1}_{0,n,p}\Psi^{k+1}_{n+1,p}+g^{k+1}_{1,n,p}\Psi^{k+1}_{n,p}+\sum_{j=2}^{n+1}g^{k+1}_{j,n,p}\Psi^{k+1}_{n-j+1,p})+$$

$$c^{k+1}_{-,n,p}(g^{k+1}_{0,n,p}\Psi^{k+1}_{n-1,p}+g^{k+1}_{1,n,p}\Psi^{k+1}_{n,p}+\sum_{j=2}^{N_x-n+1}g^{k+1}_{j,n,p}\Psi^{k+1}_{n+j-1,p})]+$$

$$\frac{1}{h_y^{\alpha^{k+1}_{n,p}}}[(c^{k+1}_{+,n,p}(g^{k+1}_{0,n,p}\Psi^{k+1}_{n,p+1}+g^{k+1}_{1,n,p}\Psi^{k+1}_{n,p}+\sum_{j=2}^{p+1}g^{k+1}_{j,n,p}\Psi^{k+1}_{n,p-j+1})+$$

$$c^{k+1}_{-,n,p}(g^{k+1}_{0,n,p}\Psi^{k+1}_{n,p-1}+g^{k+1}_{1,n,p}\Psi^{k+1}_{n,p}+\sum_{j=0}^{N_y-p+1}g^{k+1}_{j,n,p}\Psi^{k+1}_{n,p+j-1})]]=(2-2i\Delta\frac{1}{h}V^{k}_{n,p})\Psi^{k}_{n,p}$$

$$-i\frac{\Delta C_{\alpha_{n,p}^k}}{h}[\frac{1}{h_x^{\alpha_{n,p}^k}}[(c_{+,n,p}^k(g_{0,n,p}^k\Psi_{n+1,p}^k+g_{1,n,p}^k\Psi_{n,p}^k+\sum_{j=2}^{n+1}g_{j,n,p}^k\Psi_{n-j+1,p}^k)+$$

$$c_{-,n,p}^k(g_{0,n,p}^k\Psi_{n-1,p}^k+g_{1,n,p}^k\Psi_{n,p}^k+\sum_{j=2}^{N_x-n+1}g_{j,n,p}^k\Psi_{n+j-1,p}^k)]+$$

$$\frac{1}{h_y^{\alpha_{n,p}^k}}[c_{+,n,p}^k(g_{0,n,p}^k\Psi_{n,p+1}^k+g_{1,n,p}^k\Psi_{n,p}^k+\sum_{j=2}^{p+1}g_{j,n,p}^k\Psi_{n,p-j+1}^k)+c_{-,n,p}^k(g_{0,n,p}^k\Psi_{n,p-1}^k+g_{1,n,p}^k\Psi_{n,p}^k+\sum_{j=2}^{N_y-p+1}g_{j,n,p}^k\Psi_{n,p+j-1}^k)]]$$

$$\square i\frac{\Delta C_{\alpha_{n,p}^{k+1}}}{h}[-2i\frac{h}{\Delta C_{\alpha_{n,p}^{k+1}}}\Psi_{n,p}^{k+1}+\frac{1}{h_x^{\alpha_{n,p}^{k+1}}}[(c_{+,n,p}^{k+1}(g_{0,n,p}^{k+1}\Psi_{n+1,p}^{k+1}+g_{1,n,p}^{k+1}\Psi_{n,p}^{k+1}+\sum_{j=2}^{n+1}g_{j,n,p}^{k+1}\Psi_{n-j+1,p}^{k+1})+$$

$$c_{-,n,p}^{k+1}(g_{0,n,p}^{k+1}\Psi_{n-1,p}^{k+1}+g_{1,n,p}^{k+1}\Psi_{n,p}^{k+1}+\sum_{j=2}^{N_x-n+1}g_{j,n,p}^{k+1}\Psi_{n+j-1,p}^{k+1})]+$$

$$\frac{1}{h_y^{\alpha_{n,p}^{k+1}}}[(c_{+,n,p}^{k+1}(g_{0,n,p}^{k+1}\Psi_{n,p+1}^{k+1}+g_{1,n,p}^{k+1}\Psi_{n,p}^{k+1}+\sum_{j=2}^{p+1}g_{j,n,p}^{k+1}\Psi_{n,p-j+1}^{k+1})+$$

$$c_{-,n,p}^{k+1}(g_{0,n,p}^{k+1}\Psi_{n,p-1}^{k+1}+g_{1,n,p}^{k+1}\Psi_{n,p}^{k+1}+\sum_{j=0}^{N_y-p+1}g_{j,n,p}^{k+1}\Psi_{n,p+j-1}^{k+1})]]=-i\frac{\Delta C_{\alpha_{n,p}^k}}{h}[i\frac{h}{\Delta C_{\alpha_{n,p}^k}}(2-2i\Delta\frac{1}{h}V_{n,p}^k)\Psi_{n,p}^k$$

$$\frac{1}{h_x^{\alpha_{n,p}^k}}[(c_{+,n,p}^k(g_{0,n,p}^k\Psi_{n+1,p}^k+g_{1,n,p}^k\Psi_{n,p}^k+\sum_{j=2}^{n+1}g_{j,n,p}^k\Psi_{n-j+1,p}^k)+$$

$$c_{-,n,p}^k(g_{0,n,p}^k\Psi_{n-1,p}^k+g_{1,n,p}^k\Psi_{n,p}^k+\sum_{j=2}^{N_x-n+1}g_{j,n,p}^k\Psi_{n+j-1,p}^k)]+$$

$$\frac{1}{h_y^{\alpha_{n,p}^k}}[c_{+,n,p}^k(g_{0,n,p}^k\Psi_{n,p+1}^k+g_{1,n,p}^k\Psi_{n,p}^k+\sum_{j=2}^{p+1}g_{j,n,p}^k\Psi_{n,p-j+1}^k)+c_{-,n,p}^k(g_{0,n,p}^k\Psi_{n,p-1}^k+g_{1,n,p}^k\Psi_{n,p}^k+\sum_{j=2}^{N_y-p+1}g_{j,n,p}^k\Psi_{n,p+j-1}^k)]]$$

$$\square iC_{\alpha_{n,p}^{k+1}}[-2i\frac{h}{\Delta C_{\alpha_{n,p}^{k+1}}}\Psi_{n,p}^{k+1}+\frac{1}{h_x^{\alpha_{n,p}^{k+1}}}[(c_{+,n,p}^{k+1}(g_{0,n,p}^{k+1}\Psi_{n+1,p}^{k+1}+g_{1,n,p}^{k+1}\Psi_{n,p}^{k+1}+\sum_{j=2}^{n+1}g_{j,n,p}^{k+1}\Psi_{n-j+1,p}^{k+1})+$$

$$c_{-,n,p}^{k+1}(g_{0,n,p}^{k+1}\Psi_{n-1,p}^{k+1}+g_{1,n,p}^{k+1}\Psi_{n,p}^{k+1}+\sum_{j=2}^{N_x-n+1}g_{j,n,p}^{k+1}\Psi_{n+j-1,p}^{k+1})]+$$

$$\frac{1}{h_y^{\alpha_{n,p}^{k+1}}}[(c_{+,n,p}^{k+1}(g_{0,n,p}^{k+1}\Psi_{n,p+1}^{k+1}+g_{1,n,p}^{k+1}\Psi_{n,p}^{k+1}+\sum_{j=2}^{p+1}g_{j,n,p}^{k+1}\Psi_{n,p-j+1}^{k+1})+$$

$$c_{-,n,p}^{k+1}(g_{0,n,p}^{k+1}\Psi_{n,p-1}^{k+1}+g_{1,n,p}^{k+1}\Psi_{n,p}^{k+1}+\sum_{j=0}^{N_y-p+1}g_{j,n,p}^{k+1}\Psi_{n,p+j-1}^{k+1})]]=-iC_{\alpha_{n,p}^k}[i\frac{h}{\Delta C_{\alpha_{n,p}^k}}(2-2i\Delta\frac{1}{h}V_{n,p}^k)\Psi_{n,p}^k$$

$$\frac{1}{h_x^{\alpha_{n,p}^k}}[(c_{+,n,p}^k(g_{0,n,p}^k\Psi_{n+1,p}^k+g_{1,n,p}^k\Psi_{n,p}^k+\sum_{j=2}^{n+1}g_{j,n,p}^k\Psi_{n-j+1,p}^k)+$$

$$c_{-,n,p}^k(g_{0,n,p}^k\Psi_{n-1,p}^k+g_{1,n,p}^k\Psi_{n,p}^k+\sum_{j=2}^{N_x-n+1}g_{j,n,p}^k\Psi_{n+j-1,p}^k)]+$$

$$\frac{1}{h_y^{\alpha_{n,p}^k}}[c_{+,n,p}^k(g_{0,n,p}^k\Psi_{n,p+1}^k+g_{1,n,p}^k\Psi_{n,p}^k+\sum_{j=2}^{p+1}g_{j,n,p}^k\Psi_{n,p-j+1}^k)+c_{-,n,p}^k(g_{0,n,p}^k\Psi_{n,p-1}^k+g_{1,n,p}^k\Psi_{n,p}^k+\sum_{j=2}^{N_y-p+1}g_{j,n,p}^k\Psi_{n,p+j-1}^k)]]$$

make m=j-1 ;e=j-1

$$\square iC_{\alpha_{n,p}^{k+1}}[-2i\frac{h}{\Delta C_{\alpha_{n,p}^{k+1}}}\Psi_{n,p}^{k+1}+\frac{1}{h_x^{\alpha_{n,p}^{k+1}}}[(c_{+,n,p}^{k+1}(g_{0,n,p}^{k+1}\Psi_{n+1,p}^{k+1}+g_{1,n,p}^{k+1}\Psi_{n,p}^{k+1}+\sum_{m=1}^{n}g_{m+1,n,p}^{k+1}\Psi_{n-m,p}^{k+1})+$$

$$c_{-,n,p}^{k+1}(g_{0,n,p}^{k+1}\Psi_{n-1,p}^{k+1}+g_{1,n,p}^{k+1}\Psi_{n,p}^{k+1}+\sum_{m=1}^{N_x-n}g_{m+1,n,p}^{k+1}\Psi_{n+m,p}^{k+1})]+$$

$$\frac{1}{h_y^{\alpha_{n,p}^{k+1}}}[(c_{+,n,p}^{k+1}(g_{0,n,p}^{k+1}\Psi_{n,p+1}^{k+1}+g_{1,n,p}^{k+1}\Psi_{n,p}^{k+1}+\sum_{e=1}^{p}g_{e+1,n,p}^{k+1}\Psi_{n,p-e}^{k+1})+$$

$$c_{-,n,p}^{k+1}(g_{0,n,p}^{k+1}\Psi_{n,p-1}^{k+1}+g_{1,n,p}^{k+1}\Psi_{n,p}^{k+1}+\sum_{e=1}^{N_y-p}g_{e+1,n,p}^{k+1}\Psi_{n,p+e}^{k+1})]]=-iC_{\alpha_{n,p}^{k}}[i\frac{h}{\Delta C_{\alpha_{n,p}^{k}}}(2-2i\Delta\frac{1}{h}V_{n,p}^{k})\Psi_{n,p}^{k}$$

$$\frac{1}{h_x^{\alpha_{n,p}^{k}}}[(c_{+,n,p}^{k}(g_{0,n,p}^{k}\Psi_{n+1,p}^{k}+g_{1,n,p}^{k}\Psi_{n,p}^{k}+\sum_{m=1}^{n}g_{m+1,n,p}^{k}\Psi_{n-m,p}^{k})+$$

$$c_{-,n,p}^{k}(g_{0,n,p}^{k}\Psi_{n-1,p}^{k}+g_{1,n,p}^{k}\Psi_{n,p}^{k}+\sum_{m=1}^{N_x-n}g_{m+1,n,p}^{k}\Psi_{n+m,p}^{k})]+$$

$$\frac{1}{h_y^{\alpha_{n,p}^{k}}}[c_{+,n,p}^{k}(g_{0,n,p}^{k}\Psi_{n,p+1}^{k}+g_{1,n,p}^{k}\Psi_{n,p}^{k}+\sum_{e=1}^{p}g_{e+1,n,p}^{k}\Psi_{n,p-e}^{k})+c_{-,n,p}^{k}(g_{0,n,p}^{k}\Psi_{n,p-1}^{k}+g_{1,n,p}^{k}\Psi_{n,p}^{k}+\sum_{e=1}^{N_y-p}g_{e+1,n,p}^{k}\Psi_{n,p+e}^{k})]]$$

$$\square iC_{\alpha_{n,p}^{k+1}}[-2i\frac{h}{\Delta C_{\alpha_{n,p}^{k+1}}}\Psi_{n,p}^{k+1}+\frac{1}{h_x^{\alpha_{n,p}^{k+1}}}[(c_{+,n,p}^{k+1}(g_{0,n,p}^{k+1}\Psi_{n+1,p}^{k+1}+g_{1,n,p}^{k+1}\Psi_{n,p}^{k+1}+g_{2,n,p}^{k+1}\Psi_{n-1,p}^{k+1}+\sum_{m=2}^{n}g_{m+1,n,p}^{k+1}\Psi_{n-m,p}^{k+1})+$$

$$c_{-,n,p}^{k+1}(g_{0,n,p}^{k+1}\Psi_{n-1,p}^{k+1}+g_{1,n,p}^{k+1}\Psi_{n,p}^{k+1}+g_{2,n,p}^{k+1}\Psi_{n+1,p}^{k+1}+\sum_{m=2}^{N_x-n}g_{m+1,n,p}^{k+1}\Psi_{n+m,p}^{k+1})]+$$

$$\frac{1}{h_y^{\alpha_{n,p}^{k+1}}}[(c_{+,n,p}^{k+1}(g_{0,n,p}^{k+1}\Psi_{n,p+1}^{k+1}+g_{1,n,p}^{k+1}\Psi_{n,p}^{k+1}+g_{2,n,p}^{k+1}\Psi_{n,p-1}^{k+1}+\sum_{e=2}^{p}g_{e+1,n,p}^{k+1}\Psi_{n,p-e}^{k+1})+$$

$$c_{-,n,p}^{k+1}(g_{0,n,p}^{k+1}\Psi_{n,p-1}^{k+1}+g_{1,n,p}^{k+1}\Psi_{n,p}^{k+1}+g_{2,n,p}^{k+1}\Psi_{n,p+1}^{k+1}+\sum_{e=2}^{N_y-p}g_{e+1,n,p}^{k+1}\Psi_{n,p+e}^{k+1})]]=-iC_{\alpha_{n,p}^{k}}[i\frac{h}{\Delta C_{\alpha_{n,p}^{k}}}(2-2i\Delta\frac{1}{h}V_{n,p}^{k})\Psi_{n,p}^{k}$$

$$\frac{1}{h_x^{\alpha_{n,p}^{k}}}[(c_{+,n,p}^{k}(g_{0,n,p}^{k}\Psi_{n+1,p}^{k}+g_{1,n,p}^{k}\Psi_{n,p}^{k}+g_{2,n,p}^{k}\Psi_{n-1,p}^{k}+\sum_{m=2}^{n}g_{m+1,n,p}^{k}\Psi_{n-m,p}^{k})+$$

$$c_{-,n,p}^{k}(g_{0,n,p}^{k}\Psi_{n-1,p}^{k}+g_{1,n,p}^{k}\Psi_{n,p}^{k}+g_{2,n,p}^{k}\Psi_{n+1,p}^{k}+\sum_{m=2}^{N_x-n}g_{m+1,n,p}^{k}\Psi_{n+m,p}^{k})]+$$

$$\frac{1}{h_y^{\alpha_{n,p}^{k}}}[c_{+,n,p}^{k}(g_{0,n,p}^{k}\Psi_{n,p+1}^{k}+g_{1,n,p}^{k}\Psi_{n,p}^{k}+g_{2,n,p}^{k}\Psi_{n,p-1}^{k}+\sum_{e=2}^{p}g_{e+1,n,p}^{k}\Psi_{n,p-e}^{k})+$$

$$c_{-,n,p}^{k}(g_{0,n,p}^{k}\Psi_{n,p-1}^{k}+g_{1,n,p}^{k}\Psi_{n,p}^{k}+g_{2,n,p}^{k}\Psi_{n,p+1}^{k}+\sum_{e=2}^{N_y-p}g_{e+1,n,p}^{k}\Psi_{n,p+e}^{k})]]$$

$$\Box iC_{\alpha_{n,p}^{k+1}}[-2i\frac{h}{\Delta C_{\alpha_{n,p}^{k+1}}}\Psi_{n,p}^{k+1}+\frac{1}{h_x^{\alpha_{n,p}^{k+1}}}[(c_{+,n,p}^{k+1}g_{0,n,p}^{k+1}+c_{-,n,p}^{k+1}g_{2,n,p}^{k+1})\Psi_{n+1,p}^{k+1}+(c_{+,n,p}^{k+1}g_{1,n,p}^{k+1}+c_{-,n,p}^{k+1}g_{1,n,p}^{k+1})\Psi_{n,p}^{k+1}+$$

$$(c_{+,n,p}^{k+1}g_{2,n,p}^{k+1}+c_{-,n,p}^{k+1}g_{0,n,p}^{k+1})\Psi_{n-1,p}^{k+1}+c_{+,n,p}^{k+1}\sum_{m=-n}^{-2}g_{-m+1,n,p}^{k+1}\Psi_{n+m,p}^{k+1}+c_{-,n,p}^{k+1}\sum_{m=2}^{N_x-n}g_{m+1,n,p}^{k+1}\Psi_{n+m,p}^{k+1})]+$$

$$\frac{1}{h_y^{\alpha_{n,p}^{k+1}}}[(c_{+,n,p}^{k+1}g_{0,n,p}^{k+1}+c_{-,n,p}^{k+1}g_{2,n,p}^{k+1})\Psi_{n,p+1}^{k+1}+(c_{+,n,p}^{k+1}g_{1,n,p}^{k+1}+c_{-,n,p}^{k+1}g_{1,n,p}^{k+1})\Psi_{n,p}^{k+1}+(c_{+,n,p}^{k+1}g_{2,n,p}^{k+1}+c_{-,n,p}^{k+1}g_{0,n,p}^{k+1})\Psi_{n,p-1}^{k+1}+$$

$$c_{+,n,p}^{k+1}\sum_{e=-p}^{-2}g_{-e+1,n,p}^{k+1}\Psi_{n,p+e}^{k+1})+c_{-,n,p}^{k+1}\sum_{e=2}^{N_y-p}g_{e+1,n,p}^{k+1}\Psi_{n,p+e}^{k+1})]]=-iC_{\alpha_{n,p}^{k}}[i\frac{h}{\Delta C_{\alpha_{n,p}^{k}}}(2-2i\Delta\frac{1}{h}V_{n,p}^{k})\Psi_{n,p}^{k}$$

$$\frac{1}{h_x^{\alpha_{n,p}^{k}}}[(c_{+,n,p}^{k}g_{0,n,p}^{k}+c_{-,n,p}^{k}g_{2,n,p}^{k})\Psi_{n+1,p}^{k}+(c_{+,n,p}^{k}g_{1,n,p}^{k}+c_{-,n,p}^{k}g_{1,n,p}^{k})\Psi_{n,p}^{k}+(c_{+,n,p}^{k}g_{2,n,p}^{k}+c_{-,n,p}^{k}g_{0,n,p}^{k})\Psi_{n-1,p}^{k}$$

$$+c_{+,n,p}^{k}\sum_{m=-n}^{-2}g_{-m+1,n,p}^{k}\Psi_{n+m,p}^{k})+(+c_{-,n,p}^{k}\sum_{m=2}^{N_x-p}g_{m+1,n,p}^{k}\Psi_{n+m,p}^{k})]+$$

$$\frac{1}{h_y^{\alpha_{n,p}^{k}}}[(c_{+,n,p}^{k}g_{0,n,p}^{k}+c_{-,n,p}^{k}g_{2,n,p}^{k})\Psi_{n,p+1}^{k}+(c_{+,n,p}^{k}g_{1,n,p}^{k}+c_{-,n,p}^{k}g_{1,n,p}^{k})\Psi_{n,p}^{k}+(c_{+,n,p}^{k}g_{2,n,p}^{k}+c_{-,n,p}^{k}g_{0,n,p}^{k})\Psi_{n,p-1}^{k}+$$

$$c_{+,n,p}^{k}\sum_{e=-p}^{-2}g_{-e+1,n,p}^{k}\Psi_{n,p+e}^{k})+c_{-,n,p}^{k}\sum_{e=2}^{N_y-p}g_{e+1,n,p}^{k}\Psi_{n,p+e}^{k})]]$$

Here we try to solve this problem in the finite domain $[0,L_x]\times[0,L_y]$ with boundary conditions for $t>0$,

$\Psi(0,y,t)=\Psi(L_x,y,t)=0$ i.e $\Psi_{0,p}^{k}=\Psi_{N_x,p}^{k}=0$.

$\Psi(x,0,t)=\Psi(x,L_y,t)=0$ i.e $\Psi_{n,0}^{k}=\Psi_{n,N_y}^{k}=0$

$\Psi(x,y,0)=f(x,y)$ i.e $\Psi_{n,p}^{0}=f(x_n,y_p)$

we suppose that $h_x=h_y$; $L=L_x=L_y$ and $N=N_x=N_y$

$$a_{n,p}^{k}=\frac{C_{\alpha_{n,p}^{k}}}{h_x^{\alpha_{n,p}^{k}}}(c_{+,n,p}^{k}g_{2,n,p}^{k}+c_{-,n,p}^{k}g_{0,n,p}^{k})$$

$$b_{n,p}^{k}=\frac{C_{\alpha_{n,p}^{k}}}{h_x^{\alpha_{n,p}^{k}}}(c_{+,n,p}^{k}g_{0,n,p}^{k}+c_{-,n,p}^{k}g_{2,n,p}^{k})$$

$$d_{n,p}^{k}=\frac{C_{\alpha_{n,p}^{k}}}{h_x^{\alpha_{n,p}^{k}}}(c_{+,n,p}^{k}g_{1,n,p}^{k}+c_{-,n,p}^{k}g_{1,n,p}^{k}-i\frac{2hh_x^{\alpha_{n,p}^{k}}}{\Delta})$$

$$c_{n,p}^{k}=\frac{C_{\alpha_{n,p}^{k}}}{h_x^{\alpha_{n,p}^{k}}}(c_{+,n,p}^{k}g_{1,n,p}^{k}+c_{-,n,p}^{k}g_{1,n,p}^{k}+i\frac{2hh_x^{\alpha_{n,p}^{k}}}{\Delta}(1-i\Delta\frac{1}{h}V_{n,p}^{k})$$

Hence ,fanally we have

$$
\begin{aligned}
&[[b_{n,p}^{k+1}\Psi_{n+1,p}^{k+1}+d_{n,p}^{k+1}\Psi_{n,p}^{k+1}+a_{n,p}^{k+1}\Psi_{n-1,p}^{k+1}+\frac{1}{h_x^{\alpha_{n,p}^{k+1}}}c_{+,n,p}^{k+1}\sum_{m=-n}^{-2}g_{-m+1,n,p}^{k+1}\Psi_{n+m,p}^{k+1}+\frac{1}{h_x^{\alpha_{n,p}^{k+1}}}c_{-,n,p}^{k+1}\sum_{m=2}^{N_x-n}g_{m+1,n,p}^{k+1}\Psi_{n+m,p}^{k+1})]+\\
&[b_{n,p}^{k+1}\Psi_{n,p+1}^{k+1}+d_{n,p}^{k+1}\Psi_{n,p}^{k+1}+a_{n,p}^{k+1}\Psi_{n,p-1}^{k+1}+\frac{1}{h_y^{\alpha_{n,p}^{k+1}}}c_{+,n,p}^{k+1}\sum_{e=-p}^{-2}g_{-e+1,n,p}^{k+1}\Psi_{n,p+e}^{k+1})+c_{-,n,p}^{k+1}\frac{1}{h_y^{\alpha_{n,p}^{k+1}}}\sum_{e=2}^{N_y-p}g_{e+1,n,p}^{k+1}\Psi_{n,p+e}^{k+1})]]=\\
&-[[b_{n,p}^{k}\Psi_{n+1,p}^{k}+c_{n,p}^{k}\Psi_{n,p}^{k}+a_{n,p}^{k}\Psi_{n-1,p}^{k}+c_{+,n,p}^{k}\frac{1}{h_x^{\alpha_{n,p}^{k}}}\sum_{m=-n}^{-2}g_{-m+1,n,p}^{k}\Psi_{n+m,p}^{k})+(+c_{-,n,p}^{k}\frac{1}{h_x^{\alpha_{n,p}^{k}}}\sum_{m=2}^{N_x-n}g_{m+1,n,p}^{k}\Psi_{n+m,p}^{k})]+\\
&[b_{n,p}^{k}\Psi_{n,p+1}^{k}+c_{n,p}^{k}\Psi_{n,p}^{k}+a_{n,p}^{k}\Psi_{n,p-1}^{k}+c_{+,n,p}^{k}\frac{1}{h_y^{\alpha_{n,p}^{k}}}\sum_{e=-p}^{-2}g_{-e+1,n,p}^{k}\Psi_{n,p+e}^{k})+c_{-,n,p}^{k}\frac{1}{h_y^{\alpha_{n,p}^{k}}}\sum_{e=2}^{N_y-p}g_{e+1,n,p}^{k}\Psi_{n,p+e}^{k})]]
\end{aligned}
\tag{3.5}
$$

Scheme (3.5) with the boundary conditions can be written after some simplification in the matrix form as

$$A^{k+1}\Psi^{k+1}=B^{k}\Psi^{k} \tag{3.6}$$

such that the vector Ψ^k is:

$$\Psi^{k}=[\Psi_{11}^{k};.........;\Psi_{1N_y-1}^{k};..........;\Psi_{N_x-1}^{k}1...........\Psi_{N_x-1N_y-1}]^{T} \tag{3.7}$$

If we take $N_x=N_y=N$.

For $1\le i,\, j\le N-1$

$$
E_i=\begin{bmatrix}
b_{i,1}^{k+1} & 0 & 0 & 0 & & 0\\
0 & & & & & \\
0 & & b_{i,3}^{k+1} & 0 & & 0\\
0 & & & & 0 & \\
0 & & 0 & & & 0\\
0 & & 0 & 0 & & b_{i,N-1}^{k+1}
\end{bmatrix}
$$

$$
D_i=\begin{bmatrix}
a_{i,1}^{k+1} & 0 & 0 & & & 0\\
0 & a_{i,2}^{k+1} & 0 & & & \\
 & 0 & \bullet & 0 & & 0\\
0 & & 0 & \bullet & 0 & \\
0 & & & 0 & \bullet & 0\\
0 & 0 & 0 & & 0 & a_{i,N-1}^{k+1}
\end{bmatrix}
$$

$$
A_i^{k+1}=\begin{bmatrix}
2d_{i,1}^{k+1} & b_{i,1}^{k+1} & c_{-,i,1}^{k+1}\frac{1}{h_y^{\alpha_{i,1}^{k+1}}}g_{3,i,1}^{k+1} & c_{-,i,1}^{k+1}\frac{1}{h_y^{\alpha_{i,1}^{k+1}}}g_{4,i,1}^{k+1} & \cdots\cdots & \cdots\cdots & c_{-,i,1}^{k+1}\frac{1}{h_y^{\alpha_{1i}^{k+1}}}g_{N-2,i,1}^{k+1} & c_{-,i,1}^{k+1}\frac{1}{h_y^{\alpha_{i,1}^{k+1}}}g_{N-1,i,1}^{k+1} \\
a_{i,2}^{k+1} & 2d_{i,2}^{k+1} & b_{i,2}^{k+1} & \bullet & \bullet & \cdots\cdots & \cdots\cdots & c_{-,i,1}^{k+1}\frac{1}{h_y^{\alpha_{1,2}^{k+1}}}g_{N-2,i,2}^{k+1} \\
\frac{1}{h_y^{\alpha_{i,3}^{k+1}}}c_{+,i,3}^{k+1}g_{3,i,3}^{k+1} & a_{i,3}^{k+1} & + & \circ & c_{-,i,j-1}^{k+1}\frac{1}{h_y^{\alpha_{i,j-1}^{k+1}}}g_{3,i,j-1}^{k+1} & c_{-,i,j-1}^{k+1}\frac{1}{h_y^{\alpha_{i,j-1}^{k+1}}}g_{4,i,j-1}^{k+1} & \cdots\cdots & \bullet \\
\frac{1}{h_y^{\alpha_{i,4}^{k+1}}}c_{+,i,4}^{k+1}g_{4,i,4,}^{k+1} & \frac{1}{h_y^{\alpha_{i,4}^{k+1}}}c_{+,i,4}^{k+1}g_{3,i,4}^{k+1} & \circ & + & b_{i,j}^{k+1} & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \circ & + & \circ & \bullet & c_{-,i,N-4}^{k+1}\frac{1}{h_y^{\alpha_{i,j-1}^{k+1}}}g_{4,i,N-4}^{k+1} \\
\bullet & \bullet & \bullet & \bullet & \circ & + & \circ & c_{-,i,N-3}^{k+1}\frac{1}{h_y^{\alpha_{i,N-3}^{k+1}}}g_{3,i,N-3}^{k+1} \\
\bullet & \bullet & \bullet & \bullet & \bullet & \circ & + & b_{i,N-2}^{k+1} \\
\frac{1}{h_y^{\alpha_{i,N-1}^{k+1}}}c_{+,i,N-1}^{k+1}g_{N-1,i,N-1,}^{k+1} & \frac{1}{h_y^{\alpha_{i,N-1}^{k+1}}}c_{+,i,N-1}^{k+1}g_{N-2,i,4}^{k+1} & \bullet & \bullet & \frac{1}{h_y^{\alpha_{i,N-1}^{k+1}}}c_{+,i,N-1}^{k+1}g_{4,i,N-1,}^{k+1} & \frac{1}{h_y^{\alpha_{i,N-1}^{k+1}}}c_{+,i,N-1}^{k+1}g_{3,i,N-1}^{k+1} & a_{i,N-1}^{k+1} & 2d_{i,N-1}^{k+1}
\end{bmatrix}
$$

$$C_{i,j}^{-}=\begin{bmatrix} \frac{1}{h_x^{\alpha_{i,1}^{k+1}}}c_{-i,1}^{k+1}g_{j,i,1}^{k+1} & 0 & & 0 & & 0 \\ 0 & \frac{1}{h_x^{\alpha_{i,2}^{k+1}}}c_{-i,2}^{k+1}g_{j,i,2}^{k+1} & & 0 & & \\ & & & & & 0 \\ & 0 & \frac{1}{h_x^{\alpha_{i,3}^{k+1}}}c_{-i,1}^{k+1}g_{j,i,3}^{k+1} & 0 & & \\ & & & \bullet & & \\ 0 & 0 & & 0 & \bullet & \\ 0 & & & 0 & \bullet & 0 \\ 0 & & & & & \frac{1}{h_x^{\alpha_{i,N-1}^{k+1}}}c_{-i,1}^{k+1}g_{j,i,N-1}^{k+1} \end{bmatrix}$$

$$C_{i,j}^{+}=\begin{bmatrix} \frac{1}{h_x^{\alpha_{i,1}^{k+1}}}c_{+,i,1}^{k+1}g_{j,i,1}^{k+1} & 0 & & 0 & & 0 \\ 0 & \frac{1}{h_x^{\alpha_{i,2}^{k+1}}}c_{+,i,2}^{k+1}g_{j,i,2}^{k+1} & & 0 & & \\ & & & & & 0 \\ & 0 & \frac{1}{h_x^{\alpha_{i,3}^{k+1}}}c_{+,i,1}^{k+1}g_{j,i,3}^{k+1} & 0 & & \\ & & & \bullet & & \\ 0 & 0 & & 0 & \bullet & \\ 0 & & & 0 & \bullet & 0 \\ 0 & & & & & \frac{1}{h_x^{\alpha_{i,N-1}^{k+1}}}c_{+,i,1}^{k+1}g_{j,i,N-1}^{k+1} \end{bmatrix}$$

$$D_i=\begin{bmatrix} a_{i,1}^{k+1} & 0 & 0 & & & 0 \\ 0 & a_{i,2}^{k+1} & 0 & & & \\ & 0 & \bullet & 0 & & 0 \\ 0 & & 0 & \bullet & 0 & \\ 0 & & & 0 & \bullet & 0 \\ 0 & 0 & 0 & & 0 & a_{i,N-1}^{k+1} \end{bmatrix}$$

hence

$$A^{k+1}=\begin{bmatrix} A_1 & E_1 & C_{13}^{-} & C_{14}^{-} & \bullet & C_{1N-1}^{-} \\ D_2 & A_2 & E_2 & C_{23}^{-} & \bullet & \bullet \\ C_{33}^{+} & D_3 & \bullet & \bullet & \bullet & C_{N-44}^{-} \\ C_{44}^{+} & C_{43}^{+} & \bullet & \bullet & \bullet & C_{N-33}^{-} \\ \bullet & \bullet & \bullet & \bullet & \bullet & E_{N-2} \\ C_{N-1N-1}^{+} & \bullet & C_{N-14}^{+} & C_{N-13}^{+} & D_{N-1} & A_{N-1} \end{bmatrix} \quad (3.8)$$

$$B^k = -\begin{bmatrix} B_1 & E_1 & C_{13}^- & C_{14}^- & \bullet & C_{1N-1}^- \\ D_2 & B_2 & E_2 & C_{23}^- & \bullet & \bullet \\ C_{33}^+ & D_3 & \bullet & \bullet & \bullet & C_{N-44}^- \\ C_{44}^+ & C_{43}^+ & \bullet & \bullet & \bullet & C_{N-33}^- \\ \bullet & \bullet & \bullet & \bullet & \bullet & E_{N-2} \\ C_{N-1N-1}^+ & \bullet & C_{N-14}^+ & C_{N-13}^+ & D_{N-1} & B_{N-1} \end{bmatrix} \quad (3.9)$$

In B^k , We replace k+1 by k in $C_{i,j}^-, C_{i,j}^+, E_i$, D_i and $B_i = A_i$ but we replace $d_{i,j}^{k+1}$ by $c_{i,j}^k$

4 Numerical simulations

To demonstrate the effectiveness of the method, we present one numerical examples, and, we will compare our numerical results with those obtained using the exact solution

4.1 Example

Consider the space fractional Schrodinger equation with the quantum Riesz-Feller derivative with potential *V*. see [3]

$$\frac{\partial \Psi(x,y,t)}{\partial t} = -i(D_{\theta(x,y,t)|x}^{\alpha(x,y,t)} + D_{\theta(x,y,t)|y}^{\alpha(x,y,t)})\Psi(x,y,t) - V(x,y,t)\Psi(x,y,t) \quad (4.1)$$

$$0 < x, y < 2\pi \ ; \ 0 < t \le 1 \ ; \ 1 < \alpha(x,y,t) \le 2 \text{ and } \alpha(x,y,t) = 1.5 + e^{-(xyt)^2 - 1} \ ; \theta(x,y,t) = 0,2$$

$$v(x,y,t) = \frac{3}{2} - 2\frac{\sin(x + y - \frac{\theta(x,y,t)\pi}{2})}{\sin x \sin y}$$

with a initial condition :

$$\Psi(x,y,0) = \sin x \sin y$$

4.2 Remark

- It's reduce form of Schrodinger equation
- For a discretization we take N=10 and $h_x = h_y$

4.3 Progamming under Matlab

```
L=input('simlation of x and y L=')
N=input('discret of Nx and Ny  N=')
deltax=L/N
deltay=deltax
T=input('similation of t T=')
deltat=(T/L)*deltax
teta=input('skewness teta=')
k0=input('valeur de k pour la representation k0=')
K=N-1

for n=1:N-1
   for p=1:N-1

   alp=1.5+0.5*(1/exp(1))
c10=(sin(((alp-teta)*(pi/2)))/sin(alp*pi))
c20=(sin(((alp+teta)*(pi/2)))/sin(alp*pi))
cp1=1;
V0(n,p)=3/2-2*(sin((deltax*n+deltay*p)-(teta*pi/2)))/sin(deltax*n)*sin(deltay*p)
   end
end
for j=1:N-1
```

```
for n=1:N-1
 for p=1:N-1
g0(j,n,p)=(1-(1+alp)/j)*cp1;
       cp1=g0(j,n,p);
 end
 end
end
for n=1:N-1
for p=1:N-1
g1=cp1
a0(n,p)=(c10*g0(2,n,p)+c20*g1)/(deltax)^alp
b0(n,p)=(c10*g1+c20*g0(2,n,p))/(deltax)^alp
d0(n,p)=(c10*g0(1,n,p)+c20*g0(1,n,p)-(2*1i*((deltax).^alp))/(deltat))/(deltax)^alp
c0(n,p)=(c10*g0(1,n,p)+c20*g0(1,n,p)+2*1i*((deltax).^alp/deltat)*(1-1i*deltat*V0(n,p)))/(deltax)^alp
   end
end
beta10=zeros(N-1,N-1,N-1)
beta20=zeros(N-1,N-1,N-1)
E0=zeros(N-1,N-1,N-1)
E0=zeros(N-1,N-1,N-1)

for i=1:N-1
   M= diag(b0(i,:))
       E0(:,:,i)=M
       D0(:,:,i)=diag(a0(i,:))



end
for i=1:N-1
    for j=1:N-1
       for p=1:N-1
       beta10(p,j,i)=(c10*g0(j,i,p))/deltax^alp
       beta20(p,j,i)=(c20*g0(j,i,p))/deltax^alp
       end
    end
end

C10=zeros(N-1,N-1,N-1,N-1)
C20=zeros(N-1,N-1,N-1,N-1)

for i=1:N-1
     for j=1:N-1
         C10(:,:,i,j)=diag(beta10(:,j,i))
         C20(:,:,i,j)=diag(beta20(:,j,i))

     end
end
for i=1:N-1
    s(:,i,1)=2*c0(i,:)%dim=N-1
    v(:,i,2)=b0(i,1:N-2)%dim=N-2
    w(:,i,2)=a0(i,2:N-1)

end
for i=1:N-1
   for j=3:N-1
       r(1:N-j,i,j)=beta20(1:N-j,j,i)%dim=N-j
       t(j:N-1,i,j)=beta10(j:N-1,j,i)%dim=N-j
   end
end
```

```
B1=zeros(N-1,N-1,N-1)
B0=zeros((N-1)^2,(N-1)^2)
for i=1:N-1
    B1(:,:,i)=diag(s(:,i,1))+diag (v(:,i,2),1)+diag(w(:,i,2),-1)
 for   j=3:N-1

    B1(:,:,i)=B1(:,:,i)+diag(r(1:N-j,i,j),j-1)+diag(t(j:N-1,i,j),-(j-1))
 end
end
for i=1:N-1:(N-1)*(N-2)+1
   B0(i:i+N-2,i:i+N-2)=B1(:,:,((i-1)/(N-1))+1)

end
for i=1:N-1:(N-1)*(N-3)+1
  B0(i:i+N-2,i+N-1:i+N-1+N-2)=E0(:,:,((i-1)/(N-1))+1)

end
for i=1:N-1:(N-1)*(N-3)+1
   B0(i+N-1:i+N-1+N-2,i:i+N-2) =D0(:,:,((i-1)/(N-1))+1)

end
for i=1:N-1:(N-1)*(N-4)+1
   for j=i+2*(N-1):(N-1):(N-2)*(N-1)+1
       C20ij=C20(:,:,((i-1)/(N-1))+1,((j-1)/(N-1))+1)

        B0(i:i+N-2,j:j+N-2)=C20ij
   end
end
for j=1:N-1:(N-1)*(N-4)+1
    for i=j+2*(N-1):N-1:(N-1)*(N-2)+1
         C10ij=C10(:,:,((i-1)/(N-1))+1,((j-1)/(N-1))+1)
        B0(i:i+N-2,j:j+N-2)=C10ij
   end
end
for k=1:K
for n=1:N-1
    for p=1:N-1

    alp(n,p,k)=1.5+0.5*(1/exp(n*deltax*p*deltay*deltat*k+1))
c1(n,p,k)=(sin(((alp(n,p,k)-teta)*(pi/2)))/sin(alp(n,p,k)*pi))
c2(n,p,k)=(sin(((alp(n,p,k)+teta)*(pi/2)))/sin(alp(n,p,k)*pi))
cp1=1
V(n,p,k)=3/2-2*(sin((deltax*n+deltay*p)-(teta*pi/2)))/sin(deltax*n)*sin(deltay*p)
    end
end
end
for k=1:K
for j=1:N-1
 for n=1:N-1
 for p=1:N-1
g(j,n,p,k)=(1-(1+alp(n,p,k)/j))*cp1;
        cp1=g(j,n,p,k);

 end
 end
end
end
for k=1:K
 for n=1:N-1
 for p=1:N-1
```

```
g1=cp1
a(n,p,k)=(c1(n,p,k)*g(2,n,p,k)+c2(n,p,k)*g1)/(deltax)^alp(n,p,k)
b(n,p,k)=(c1(n,p,k)*g1+c2(n,p,k)*g(2,n,p,k))/(deltax)^alp(n,p,k)
d(n,p,k)=(c1(n,p,k)*g(1,n,p,k)+c2(n,p,k)*g(1,n,p,k)-(2*1i*((deltax).^alp(n,p,k))/(deltat)))/(deltax)^alp(n,p,k)
c(n,p,k)=(c1(n,p,k)*g(1,n,p,k)+c2(n,p,k)*g(1,n,p,k)+2*1i*((deltax).^alp(n,p,k)/deltat)*(1-
1i*deltat*V(n,p,k)))/(deltax)^alp(n,p,k)
 end
 end
end

 beta1=zeros(N-1,N-1,N-1,N-1)
beta2=zeros(N-1,N-1,N-1,N-1)
for k=1:K
for i=1:N-1

        E(:,:,i,k)=diag(b(i,:,k))
        D(:,:,i,k)=diag(a(i,:,k))


end
end
for k=1:K
for i=1:N-1
    for j=1:N-1
        for p=1:N-1
        beta1(p,j,i,k)=(c1(i,p,k)*g(j,i,p,k))/deltax^alp(i,p,k)
        beta2(p,j,i,k)=(c2(i,p,k)*g(j,i,p,k))/deltax^alp(i,p,k)
        end
    end
end
end

C1=zeros(N-1,N-1,N-1,N-1,N-1)
C2=zeros(N-1,N-1,N-1,N-1,N-1)
for k=1:K
for i=1:N-1
      for j=1:N-1
          C1(:,:,i,j,k)=diag(beta1(:,j,i,k))
          C2(:,:,i,j,k)=diag(beta2(:,j,i,k))

      end
end
end
for k=1:K
for i=1:N-1
    s(:,i,1,k)=2*c(i,:,k)%dim=N-1
    s1(:,i,1,k)=2*d(i,:,k)
    v(:,i,2,k)=b(i,1:N-2,k)%dim=N-2
     w(:,i,2,k)=a(i,2:N-1,k)%dim=N-2
end
end
for k=1:K
for i=1 : N-1
   for j=3:N-1
        r(1:N-j,i,j,k)=beta2(1:N-j,j,i,k)%dim=N-j
        t(j:N-1,i,j,k)=beta1(j:N-1,j,i,k)%dim=N-j

   end
end
end
```

```
B1=zeros(N-1,N-1,N-1,N-1)
B=zeros((N-1)^2,(N-1)^2,N-1)
A1=zeros(N-1,N-1,N-1,N-1)
A=zeros((N-1)^2,(N-1)^2,N-1)
for k=1:K
for i=1:N-1
   B1(:,:,i,k)= diag(s(:,i,1,k))+diag (v(:,i,2,k),1)+diag(w(:,i,2,k),-1)
   A1(:,:,i,k)= diag(s1(:,i,1,k))+diag (v(:,i,2,k),1)+diag(w(:,i,2,k),-1)

 for   j=3:N-1

      B1(:,:,i,k)=B1(:,:,i,k)+diag(r(1:N-j,i,j,k),j-1)+diag(t(j:N-1,i,j,k),-(j-1))
       A1(:,:,i,k)=A1(:,:,i,k)+diag(r(1:N-j,i,j,k),j-1)+diag(t(j:N-1,i,j,k),-(j-1))
 end
end
end
for k=1:K
for i=1:N-1:(N-1)*(N-2)+1
 B(i:i+N-2,i:i+N-2,k)=B1(:,:,((i-1)/(N-1))+1,k)
 A(i:i+N-2,i:i+N-2,k)=A1(:,:,((i-1)/(N-1))+1,k)
end
end
for k=1:K
for i=1:N-1:(N-1)*(N-3)+1
  B(i:i+N-2,i+N-1:i+N-1+N-2,k)= E(:,:,((i-1)/(N-1))+1,k)
  A(i:i+N-2,i+N-1:i+N-1+N-2,k)= E(:,:,((i-1)/(N-1))+1,k)
end
end
for k=1:K
for i=1:N-1:(N-1)*(N-3)+1
  B(i+N-1:i+N-1+N-2,i:i+N-2,k)=D(:,:,((i-1)/(N-1))+1,k)
  A(i+N-1:i+N-1+N-2,i:i+N-2,k)=D(:,:,((i-1)/(N-1))+1,k)
end
end
for k=1:K
for i=1:N-1:(N-1)*(N-4)+1
   for j=i+2*(N-1):(N-1):(N-2)*(N-1)+1
        B(i:i+N-2,j:j+N-2,k) =C2(:,:,((i-1)/(N-1))+1,((j-1)/(N-1))+1,k)
        A(i:i+N-2,j:j+N-2,k) =C2(:,:,((i-1)/(N-1))+1,((j-1)/(N-1))+1,k)

   end
end
end
for k=1:K
for j=1:N-1:(N-1)*(N-4)+1
    for i=j+2*(N-1):N-1:(N-1)*(N-2)+1
       B(i:i+N-2,j:j+N-2,k)=C1(:,:,((i-1)/(N-1))+1,((j-1)/(N-1))+1,k)
       A(i:i+N-2,j:j+N-2,k)=C1(:,:,((i-1)/(N-1))+1,((j-1)/(N-1))+1,k)

    end
end
end

for i=1:N-1
for j=1:N-1
        V0(i,j)=sin(i*deltax)*sin(j*deltay);
end
end
```

```
 U0=reshape(V0',[(N-1)^2 1])
for k=1:K-1

    U(:,1)= inv(A(:,:,1))*(B0*U0);

   U(:,k+1)=inv(A(:,:,k+1))*(B(:,:,k)*U(:,k))
end
W=reshape(U(:,k0),[N-1 N-1])
V(:,:,k0)=W'
T=zeros(N+1,N+1)
T(2:N,2:N)=W'
for i=1:N+1
 for   j=1:N+1
   T1(i,j)=abs(T(i,j))
   T2(i,j)=real(T(i,j))
   T3(i,j)=imag(T(i,j))
 end

[i,j]=meshgrid(1:1:N+1,1:1:N+1)
 figure(1)
figure(1)
surf(i,j,T1)
figure(2)
surf(i,j,T2)
figure(3)
surf(i,j,T3)
```

4.4 Results of the programming/ Graphicals representations

Figure 1 : Re $\Psi(x, y, 0.1)$

Figure 2 : Im $\Psi(x, y, 0.1)$

Figure 3 : $|\Psi(x, y, 0.1)|$

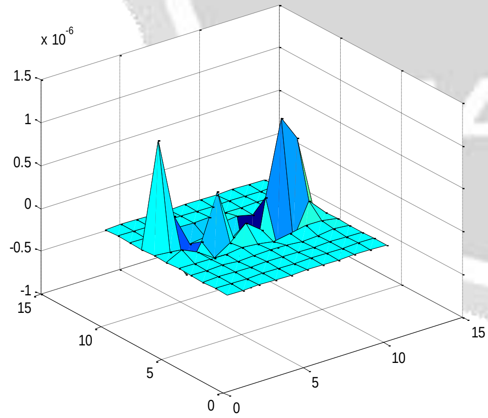

Figure 1

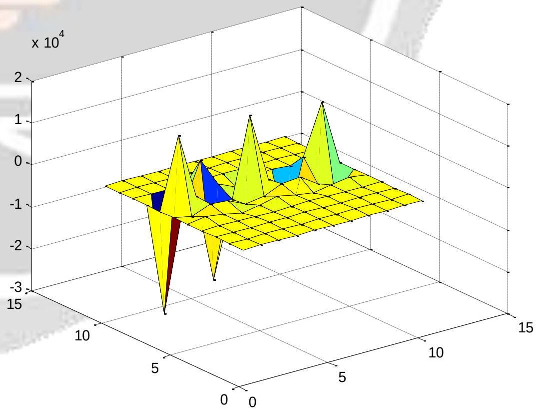

Figure 2

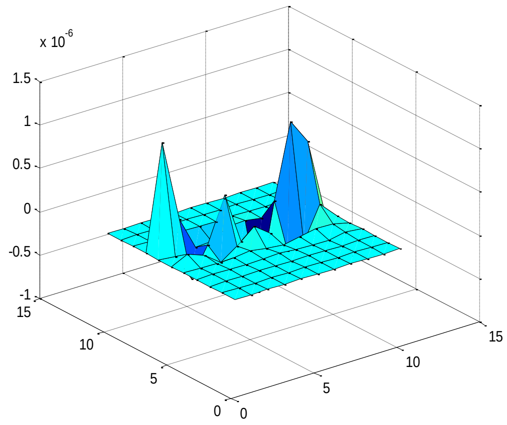

Figure 3

5 Conclusion

In this paper, the shifted Grünwald-letnikov is used to solve the problem . An axample is given. There may be other questions to answer that we may see next time All results in this paper are obtained using MATLAB (2013a).

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