One Numerical Simulation for the Variable Order Fractional Schrodinger Equation with the Quantum Riesz-Feller Derivative

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Abstract

In this paper the space variable-order fractional Schrodinger equation is studied numerically, where the variable-order fractional derivative is described here in the sense of the quantum Riesz-Feller definition. The proposed numerical method is analogue of the Crank–Nicholson method and right-shifted formula of Grunwald-Letnikov

***Keywords**: Fractional Schrodinger equation, quantum Riesz-Feller definition, right-shifted formula of Grunwald-Letnikov, Crank–Nicholson method*

1. Introduction

The concept of fractional derivatives is by no means new. In fact, they are almost as old as their more familiar integer order counterparts . Until recently, however, fractional derivatives have been successfully applied to problems in system biology [4], physics [5–9], chemistry and biochemistry [10], hydrology [11–14], and finance [15–18]. These new fractional-order models are more adequate than the previously used integer-order models, because fractional-order derivatives and integrals enable the description of the memory and hereditary properties of different substances [27]. This is the most significant advantage of the fractional-order models in comparison with integer-order models, in which such effects are neglected. In the area of physics, fractional space derivatives are used to model anomalous diffusion or dispersion, where a particle spreads at a rate inconsistent with the classical Brownian motion model [7]. In particular, Laskin [20–2 3] constructed the space fractional Schrodinger equation in form (1.1) as a generalization of the classical Schrodinger equation, obtained by replacing the second order space derivative by a Riesz fractional derivative.

$$i\frac{h}{2\pi}\frac{\partial\Psi(r,t)}{\partial t}=C_\alpha(m)(-\Delta)^{\frac{\alpha}{2}}\Psi(r,t)+V(r,t)\Psi(r,t)\quad t\geq 0, r\in\mathbb{R}\quad (1.1)$$

for the wave function Ψ of a quantum particle with the mass m that moves in a potential field with the potential *V*. Where h is the Plank constant, $C_\alpha(m)$ is a positive constant which equals $\frac{h^2}{2m}$ for $\alpha=2$ and $(-\Delta)^{\frac{\alpha}{2}}$ was called the quantum Riesz fractional derivative of order α . In the mathematical literature, $(-\Delta^{\frac{\alpha}{2}})$ is usually referred to as the fractional Laplacian. For $\alpha=2$, the quantum Riesz fractional derivative becomes the negative Laplace operator $-\Delta$ and eq (1.1) is reduced to the classical Schrodinger equation for a quantum particle with the mass m that moves in a potential field with the potential V

Many papers using several analytical and numerical methods have dealt with the space-fractional and space time-fractional Schrodinger equations with some specific potential fields including zero potential (free particle), the δ -potential, the infinite potential well, the Coulomb potential, and a rectangular barrier. In 2013, Al-Saqabi et al [6] solved the fractional Schrodinger equation with the quantum Riesz-Feller derivative for a particle that moves in a potential field in terms of the Fox *H*-function. Atangana et al. [28] solved the space variable-order fractional Schrodinger equation using Crank-Nicholson scheme, they used the Caputo variable-order differential operator

The main aim of this paper is to introduce numerical study of the variable-order fractional Schrodinger equation with the quantum Riesz-Feller derivative using the right schifted formula Grunwald-Letnikov[1] [2].

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This paper is structured as follows: in the next section we introduce some definitions on fractional calculus and some properties of non-standard discretization. Section 3 is devoted to discretization.. In Section 4, some numerical treatments are established with their results.
Concluding are given in Section 5.

2 Preliminaries and notations

In the following we give some preliminary results which are needed in subsequent sections of this paper.

2.1 Fractional calculus definitions

In the labels, many different definitions of the fractional derivatives were introduced (see e. [12, 15, 19, 24]). The time-fractional derivatives are often given in the Caputo,Riemann-Liouville, or Grunwald-Letnikov sense. As to the space-fractional derivative, it is usually defined as an operator inverse to the Riesz potential and is referred to as the Riesz fractional derivative. Podlubny concludes ([15]) that "the complete theory of fractional differential equations,especially the theory of boundary value problems for fractional differential equations,can be developed only with the use of both left and right derivatives. So the spatial derivatives discussed in this paper are the fractional Riesz-Feller potential operator, which includes the left and right Riemann-Liouville fractional derivatives.

For $0 < \alpha < 2$, $\alpha \neq 1$ and $|\theta| \le \{\alpha, 2-\alpha\}$, the quantum Riesz-Feller fractional derivative D_θ^α was represented in the following form (see [6, 19]):

Definition 2.1.

$$D_\theta^\alpha u(x) = (c_+ D_+^\alpha + c_- D_-^\alpha) u(x) \qquad (2.1)$$

where the coefficients $c\pm$ are given by

$$c_+ = c_+(\alpha,\theta) = \frac{\sin((\alpha-\theta)\frac{\pi}{2})}{\sin(\alpha\pi)} \qquad c_- = c_-(\alpha,\theta) = \frac{\sin((\alpha+\theta)\frac{\pi}{2})}{\sin(\alpha\pi)} \qquad (2.2)$$

$$(D_+^\alpha u)(x) = (\frac{d}{dx})^n (I_+^{n-\alpha} f)(x), \qquad (D_-^\alpha u)(x) = (-\frac{d}{dx})^n (I_-^{n-\alpha} u)(x) \qquad (2.3)$$

are the left-side and right-side Riemann-Liouville fractional derivatives with $x \in \mathbb{R}$ and $\alpha > 0, n-1 < \alpha \le n, n = 1,2$ In expressions (2.3) the fractional operators $I_\pm^{n-\alpha}$ are defined as the left- and right-side of Weyl fractional integrals, which given by

$$(I_+^\alpha u)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{u(\xi)}{(x-\xi)^{1-\alpha}}, \qquad (I_-^\alpha u)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \frac{u(\xi)}{(x-\xi)^{1-\alpha}} \qquad (2.4)$$

For $\alpha = 1$, the representation (2.1) is not valid and has to be replaced by the formula

$$D_0^1 u(x) = \left[\cos(\theta\frac{\pi}{2}) D_0^1 - \sin(\theta\frac{\pi}{2}) D \right] u(x) \qquad (2.5)$$

where D refers for the first standard derivative and the operator D_0^1 is related to the Hilbert transform as first noted by Feller in 1952 in his pioneering paper [25]

$$D_0^1 u(x) = \frac{1}{\pi}\frac{d}{dx}\int_{-\infty}^{+\infty} \frac{u(\xi)}{x-\xi} d\xi$$

We can naturally expand this definition to a variable-order quantum Riesz-Feller fractional derivative $D_{\theta(x)}^{\alpha(x)}$ for $0 < \alpha(x) < 2, \alpha(x) \neq 1$ and $|\theta(x) \le \min\{\alpha(x), 2-\alpha(x)\}|$ as following

Definition 2.2 For

$$D_{\theta(x)}^{\alpha(x)} u(x) = (c_+ D_+^{\alpha(x)} + c_- D_-^{\alpha(x)}) u(x) \qquad (2.6)$$

$$c_+ = c_+(\alpha(x),\theta(x)) = \frac{\sin((\alpha(x)-\theta(x))\frac{\pi}{2})}{\sin(\alpha(x)\pi)}, \quad c_- = c_-(\alpha(x),\theta(x)) = \frac{\sin((\alpha(x)+\theta(x))\frac{\pi}{2})}{\sin(\alpha(x)\pi)} \qquad (2.7)$$

2.2 Right-shifted formula of Grunwald-Letnikov

The shifted Grünwald formula for discretizing the two-sided fractional derivative is proposed by Meerschaert and Tadjeran [29], and it was shown that the standard (i.e., unshifted) Grünwald formula to discretize the fractional diffusion equation results in an unstable finite difference scheme regardless of whether the resulting finite difference method is an explicit or an implicit system. Hence, we discretise the Riesz-Feller fractional derivative $D_\theta^\alpha u$ by the shifted Grünwald formula [29]:

If the spatial domain is $[0, L]$ The mesh is N equals intervals of $\delta = \frac{L}{N}$ and $x_l = l\delta$ for $0 \le l \le N$, and $u(l\delta) = u_l$

$$_0D_x^\alpha u(x_l) = \frac{1}{\delta^\alpha}\sum_{j=0}^{l+1} g_j u_{l-j+1} + O(\delta) \qquad _xD_L^\alpha u(x_l) = \frac{1}{\delta^\alpha}\sum_{j=0}^{N-l+1} g_j u_{l+j-1} + O(\delta) \; [1] \quad (2.8)$$

Where the coefficients are defined by

$$g_0 = 1 \qquad g_j = (-1)^j \frac{\alpha(\alpha-1).........(\alpha-j+1)}{j!} \quad (2.9)$$

We also have $g_j = (1 - \frac{\alpha+1}{j}) g_{j-1} \qquad j = 1,........, N$ [1] (2.10)

In this paper, we consider the variable order fractional Schrodinger equation with the quantum Riesz–Feller derivative that describes the wave function Ψ of a quantum particle that moves in a potential field in one dimensional space with the potential V in the form:

$$ih\frac{\partial\Psi(x,t)}{\partial t} = C_{\alpha(x,,t)}(m) D_{\theta(x,t)}^{\alpha(x,t)} \Psi(x,t) + V(x,t)\Psi(x,t) \;,\; t > 0 \,,\; \Psi(x,0) = f(x) \; x \in [0, L] \quad (2.11)$$

In this case $g_0 = 1 \quad g_j(x) = (-1)^j \frac{\alpha(x)(\alpha(x)-1).........(\alpha(x)-j+1)}{j!}$

3 Discretization

Let $t \in [0,T]$, $\Delta = \frac{T}{M}$, $t_k = \Delta k$, $k = 0,1,........,M$ and $\Psi(x_l, t_k) = \Psi_{l,k}$. If $x \in [0, L]$

$$D_{\theta_{l,k}}^{\alpha_{l,k}}(\Psi_{l,k}) = (c_{+,l,k}\; {}_0D_x^{\alpha_{l,k}} + c_{-,l,k}\; {}_xD_L^{\alpha_{l,k}})(\Psi_{l,k}) = \frac{1}{\delta^\alpha}\left[c_{+,l,k}\sum_{j=0}^{l+1} g_{j,l,k}\Psi_{l-j+1,k} + c_{-,l,k}\sum_{j=0}^{N-l+1} g_{j,l,k}\Psi_{l+j-1,k}\right] + O(\delta) \quad (3.1)$$

With $c_{+,l,k} = c_+(x_l, t_k) \quad c_{-,l,k} = c_-(x_l, t_k) \quad \alpha_{l,k} = \alpha(x_l, t_k) \quad \theta_{l,k} = \theta(x_l, t_k)$

$$g_{j,l,k} = g_j(x_l, t_k) = (-1)^j \frac{\alpha(x_l,t_k)(\alpha(x_l,t_k)-1).........(\alpha(x_l,t_k)-j+1)}{j!} \quad (3.2)$$

Applying the usual approximation to a derivative on time and interpolation designs with extrapolation, as a result we receive analog of Crank-Nicholson method in (2.11) [2]

$$ih\frac{\partial\Psi(x_l,t_k)}{\partial t} = C_{\alpha_{l,k}}(m) D_{\theta_{l,k}}^{\alpha_{l,k}} \Psi(x_l,t_k) + V(x_l,t_k)\Psi(x_l,t_k) \quad (3.3)$$

$$ih\frac{\Psi_{l,k+1} - \Psi_{l,k}}{\Delta} = \frac{C_{\alpha_{l,k+1}} D_{\theta_{l,k+1}}^{\alpha_{l,k+1}} \Psi_{l,k+1} + C_{\alpha_{l,k}} D_{\theta_{l,k}}^{\alpha_{l,k}} \Psi_{l,k}}{2} + V_{l,k}\Psi_{l,k} \quad (3.4)$$

Where $V_{l,k} = V(x_l, t_k)$

$$\square ih\frac{\Psi_{l,k+1}-\Psi_{l,k}}{\Delta}=\frac{C_{\alpha_{l,k+1}}D_{\theta_{l,k+1}}^{\alpha_{l,k+1}}\Psi_{l,k+1}+C_{\alpha_{l,k}}D_{\theta_{l,k}}^{\alpha_{l,k}}\Psi_{l,k}}{2}+V_{l,k}\Psi_{l,k}$$

$$\square 2\Psi_{l,k+1}+i\frac{1}{h}\Delta C_{\alpha_{L,k+1}}D_{\theta_{l,k+1}}^{\alpha_{l,k=1}}\Psi_{l,k+1}=(2-2i\Delta\frac{1}{h}V_{l,k})\Psi_{l,k}-i\frac{1}{h}\Delta C_{\alpha_{l,k}}D_{\theta_{l,k}}^{\alpha_{l,k}}\Psi_{l,k}$$

$$\square 2\Psi_{l,k+1}+i\frac{\Delta C_{\alpha_{l,k+1}}}{h\delta^{\alpha_{l,k+1}}}(c_{+,l,k+1}\sum_{j=0}^{l+1}g_{j,l,k+1}\Psi_{l-j+1,k+1}+c_{-,l,k+1}\sum_{j=0}^{N-l+1}g_{j,l,k+1}\Psi_{l+j-1,k+1})=$$

$$(2-2i\Delta\frac{1}{h}V_{l,k})\Psi_{l,k}-i\frac{\Delta C_{\alpha_{l,k}}}{h\delta^{\alpha_{l,k}}}(c_{+,l,k}\sum_{j=0}^{l+1}g_{j,l,k}\Psi_{l-j+1,k}+c_{-,l,k}\sum_{j=0}^{N-l+1}g_{j,l,k}\Psi_{l+j-1,k})$$

$$\square 2\Psi_{l,k+1}+i\frac{\Delta C_{\alpha_{l,k+1}}}{h\delta^{\alpha_{l,k+1}}}(c_{+,l,k+1}(g_0\Psi_{l+1,k+1}+g_{1,l,k+1}\Psi_{l,k+1}+\sum_{j=2}^{l+1}g_{j,l,k+1}\Psi_{l-j+1,k+1})+$$

$$c_{-,l,k+1}(g_0\Psi_{l-1,k+1}+g_{1,l,k+1}\Psi_{l,k+1}+\sum_{j=2}^{N-l+1}g_{j,l,k+1}\Psi_{l+j-1,k+1}))=$$

$$(2-2i\Delta\frac{1}{h}V_{l,k})\Psi_{l,k}-i\frac{\Delta C_{\alpha_{l,k}}}{h\delta^{\alpha_{l,k}}}(c_{+,l,k}(g_0\Psi_{l+1,k}+g_{1,l,k}\Psi_{l,k}+\sum_{j=2}^{l+1}g_{j,l,k}\Psi_{l-j+1,k})+$$

$$c_{-,l,k+1}(g_0\Psi_{l-1,k}+g_{1,l,k+1}\Psi_{l,k}+\sum_{j=2}^{N-l+1}g_{j,l,k}\Psi_{l+j-1,k}))$$

$$\square i\frac{\Delta C_{\alpha_{l,k+1}}}{h\delta^{\alpha_{l,k+1}}}(-2i\frac{h\delta^{\alpha_{l,k+1}}}{\Delta C_{\alpha_{l,k+1}}}\Psi_{l,k+1}+c_{+,l,k+1}(g_0\Psi_{l+1,k+1}+g_{1,l,k+1}\Psi_{l,k+1}+\sum_{j=2}^{l+1}g_{j,l,k+1}\Psi_{l-j+1,k+1})+$$

$$c_{-,l,k+1}(g_0\Psi_{l-1,k+1}+g_{1,l,k+1}\Psi_{l,k+1}+\sum_{j=2}^{N-l+1}g_{j,l,k+1}\Psi_{l+j-1,k+1}))=$$

$$-i\frac{\Delta C_{\alpha_{l,k}}}{h\delta^{\alpha_{l,k}}}(2i\frac{h\delta^{\alpha_{l,k}}}{\Delta C_{\alpha_{l,k}}}(1-i\Delta\frac{1}{h}V_{l,k})\Psi_{l,k}+c_{+,l,k}(g_0\Psi_{l+1,k}+g_{1,l,k}\Psi_{l,k}+\sum_{j=2}^{l+1}g_{j,l,k}\Psi_{l-j+1,k})+$$

$$c_{-,l,k+1}(g_0\Psi_{l-1,k}+g_{1,l,k+1}\Psi_{l,k}+\sum_{j=2}^{N-l+1}g_{j,l,k}\Psi_{l+j-1,k}))$$

Si $\beta_{l,k}=\frac{C_{\alpha_{l,k}}}{\delta^{\alpha_{l,k}}}$

$$\square\beta_{l,k+1}[(-2i\frac{h\delta^{\alpha_{l,k+1}}}{\Delta C_{\alpha_{l,k+1}}}\Psi_{l,k+1}+c_{+,l,k+1}g_{1,l,k+1}\Psi_{l,k+1}+c_{+,l,k+1}g_0\Psi_{l+1,k+1}+c_{+,l,k+1}\sum_{j=2}^{l+1}g_{j,l,k+1}\Psi_{l-j+1,k+1})+$$

$$c_{-,l,k+1}g_0\Psi_{l-1,k+1}+c_{-,l,k+1}g_{1,l,k+1}\Psi_{l,k+1}+c_{-,l,k+1}\sum_{j=2}^{N-l+1}g_{j,l,k+1}\Psi_{l+j-1,k+1})]=$$

$$\beta_{l,k}[-2i\frac{h\delta^{\alpha_{l,k}}}{\Delta C_{\alpha_{l,k}}}(1-i\frac{1}{h}\Delta V_{l,k})\Psi_{l,k}-c_{+,l,k}g_0\Psi_{l+1,k}-c_{+,l,k}g_{1,l,k}\Psi_{l,k}-c_{+,l,k}\sum_{j=2}^{l+1}g_{j,l,k}\Psi_{l-j+1,k})-$$

$$c_{-,l,k+1}g_0\Psi_{l-1,k}-c_{-,l,k+1}g_{1,l,k+1}\Psi_{l,k}-c_{-,l,k+1}\sum_{j=2}^{N-l+1}g_{j,l,k}\Psi_{l+j-1,k}))]$$

make m=j-1,

$$\Rightarrow \beta_{l,k+1}[-2i\frac{h\delta^{\alpha_{l,k+1}}}{\Delta C_{\alpha_{l,k+1}}}\Psi_{l,k+1}+c_{+,l,k+1}g_0\Psi_{l+1,k+1}+c_{+,l,k+1}g_{1,l,k+1}\Psi_{l,k+1}+c_{+,l,k+1}g_{2,l,k+1}\Psi_{l-1,k+1}+c_{+,l,k+1}\sum_{m=2}^{l}g_{m+1,l,k+1}\Psi_{l-m,k+1}$$

$$c_{-,l,k+1}g_0\Psi_{l-1,k+1}+c_{-,l,k+1}g_{1,l,k+1}\Psi_{l,k+1}+c_{-,l,k+1}g_{2,l,k+1}\Psi_{l+1,k+1}+c_{-,l,k+1}\sum_{m=2}^{N-l}g_{m+1,l,k+1}\Psi_{l+m,k+1}]=$$

$$\beta_{l,k}[-2i\frac{h\delta^{\alpha_{l,k}}}{\Delta C_{\alpha_{l,k}}}(1-i\Delta\frac{1}{h}V_{l,k})\Psi_{l,k}-c_{+,l,k}g_0\Psi_{l+1,k}-c_{+,l,k}g_{1,l,k}\Psi_{l,k}-c_{+,l,k}g_{2,l,k}\Psi_{l-1,k}-c_{+,l,k}\sum_{m=2}^{l}g_{m+1,l,k}\Psi_{l-m,k})-$$

$$c_{-,l,k}g_0\Psi_{l-1,k}-c_{-,l,k}g_{1,l,k}\Psi_{l,k}-c_{-,l,k}g_{2,l,k}\Psi_{l+1,k}-c_{-,l,k}\sum_{m=2}^{N-l}g_{m+1,l,k+1}\Psi_{l+m,k+1}]$$

$$\Rightarrow \beta_{l,k+1}[(c_{+,l,k+1}g_{2,l,k+1}+c_{-,l,k+1}g_0)\Psi_{l-1,k+1}+(c_{+,l,k+1}g_{1,l,k+1}+c_{-,l,k+1}g_{1,l,k+1}-2i\frac{h\delta^{\alpha_{l,k+1}}}{\Delta C_{\alpha_{l,k+1}}})\Psi_{l,k+1}+$$

$$+(c_{+,l,k+1}g_0+c_{-,l,k+1}g_{2,l,k+1})\Psi_{l+1,k+1}+c_{+,l,k+1}\sum_{m=2}^{l}g_{m+1,l,k+1}\Psi_{l-m,k+1})++c_{-,l,k+1}\sum_{m=2}^{N-l}g_{m+1,l,k+1}\Psi_{l+m,k+1}]=$$

$$\beta_{l,k}[-(c_{+,l,k}g_{2,l,k}+c_{-,l,k}g_0)\Psi_{l-1,k}-(c_{+,l,k}g_{1,l,k}+c_{-,l,k}g_{1,l,k}+2i\frac{h\delta^{\alpha_{l,k}}}{\Delta C_{\alpha_{l,k}}}(1-i\Delta\frac{1}{h}V_{l,k}))\Psi_{l,k}-$$

$$(c_{+,l,k}g_0+c_{-,l,k}g_{2,l,k})\Psi_{l+1,k}-c_{+,l,k}\sum_{m=2}^{l}g_{m+1,l,k}\Psi_{l-m,k})-c_{-,l,k}\sum_{m=2}^{N-l}g_{m+1,l,k+1}\Psi_{l+m,k+1}]$$

$$a_{l,k+1}\Psi_{l-1,k+1}+d_{l,k+1}\Psi_{l,k+1}+b_{l,k+1}\Psi_{l+1,k+1}+\beta_{l,k+1}c_{+,l,k+1}\sum_{m=-l}^{-2}g_{-m+1,l,k+1}\Psi_{l+m,k+1}+\beta_{l,k+1}c_{-,l,k+1}\sum_{m=2}^{N-l}g_{m+1,l,k+1}\Psi_{l+m,k+1})=$$
$$-(a_{l,k}\Psi_{l-1,k}+c_{l,k}\Psi_{l,k}+b_{l,k}\Psi_{l+1,k}+\beta_{l,k}c_{+,l,k}\sum_{m=-l}^{-2}g_{-m+1,l,k}\Psi_{l+m,k}+\beta_{l,k}c_{-,k,l}\sum_{m=2}^{N-l}g_{m+1,l,k}\Psi_{l+m,k}) \quad (3.5)$$

where $a_{l,k}=\beta_{l,k}(c_{+,l,k}g_{2,l,k}+c_{-,l,k}g_0)$ $b_{l,k}=\beta_{l,k}(c_{+,l,k}g_0+c_{-,l,k}g_{2,l,k})$

$$d_{l,k}=\beta_{l,k}(c_{+,l,k}g_{1,l,k}+c_{-,l,k}g_{1,l,k}-2i\frac{h\delta^{\alpha_{l,k}}}{\Delta C_{\alpha_{l,k}}})$$

$$c_{l,k}=\beta_{l,k}[c_{+,l,k}g_{1,l,k}+c_{-,l,k}g_{1,l,k}+2i\frac{h\delta^{\alpha_{l,k}}}{\Delta C_{\alpha_{l,k}}}(1-i\Delta\frac{1}{h}V_{l,k})]$$

Here we try to solve this problem in the finite domain [0,L] with boundary conditions for t>0,

$\Psi(0,t)=\Psi(L,t)=0$ i.e $\Psi_{0,k}=\Psi_{N,k}=0$. $\Psi(x_l,0)=f(x_l)$ i.e $\Psi_{l,0}=f(\delta l)$ (3.6)

We write (3.5) for $l=1,........,N-1$:

$$a_{l,k+1}\Psi_{l-1,k+1}+d_{l,k+1}\Psi_{l,k+1}+b_{l,k+1}\Psi_{l+1,k+1}+\beta_{l,k+1}c_{+,l,k+1}\sum_{m=-l}^{-2}g_{-m+1,l,k+1}\Psi_{l+m,k+1}+\beta_{l,k+1}c_{-,l,k+1}\sum_{m=2}^{N-l}g_{m+1,l,k+1}\Psi_{l+m,k+1})=$$

$$-(a_{l,k}\Psi_{l-1,k}+c_{l,k}\Psi_{l,k}+b_{l,k}\Psi_{l+1,k}+\beta_{l,k}c_{+,l,k}\sum_{m=-l}^{-2}g_{-m+1,l,k}\Psi_{l+m,k}+\beta_{l,k}c_{-,k,l}\sum_{m=2}^{N-l}g_{m+1,l,k}\Psi_{l+m,k})$$

$l=1$

$$d_{1,k+1}\Psi_{1,k+1}+b_{1,k+1}\Psi_{2,k+1}+\beta_{1,k+1}c_{-,1,k+1}\sum_{m=2}^{N-1}g_{m+1,1,k+1}\Psi_{1+m,k+1}=-c_{1,k}\Psi_{1,k}-b_{1,k}\Psi_{2,k}-\beta_{l,k}c_{-,1,k}\sum_{m=2}^{N-1}g_{m+1,1,k}\Psi_{1+m,k}$$

$l=2$

$$a_{2,k+1}\Psi_{1,k+1}+d_{2,k+1}\Psi_{2,k+1}+b_{2,k+1}\Psi_{3,k+1}+\beta_{2,k+1}c_{-,2,k+1}\sum_{m=2}^{N-2}g_{m+1,2,k+1}\Psi_{2+m,k+1}=-a_{2,k+1}\Psi_{1,k}-c_{2,k}\Psi_{2,k}-b_{2,l}\Psi_{3,k}-\beta_{2,k+1}c_{-,2,k}\sum_{m=2}^{N-2}g_{m+1,2,k}\Psi_{2+m,k}$$

$l=3$

$$\beta_{3,k+1}c_{+,3,k+1}g_{3,3,k+1}\Psi_{1,k+1}+a_{3,k+1}\Psi_{2,k+1}+d_{3,k+1}\Psi_{3,k+1}+b_{3,k+1}\Psi_{4,k+1}+\beta_{3,k+1}c_{-,3,k+1}\sum_{m=2}^{N-3}g_{m+1,3,k+1}\Psi_{3+m,k+1}=$$

$$-\beta_{3,k}c_{+,3,k}g_{3,3,k}\Psi_{1,k}-a_{3,k+1}\Psi_{2,k}-\beta_{3,k}c_{3,k}\Psi_{l,k}-b_{3,k}\Psi_{4,k}-\beta_{3,k}c_{-,3,k}\sum_{m=2}^{N-3}g_{m+1,3,k+1}\Psi_{3+m,k}$$

$l=4$

$$\beta_{4,k+1}c_{+,4,k+1}g_{4,4,k+1}\Psi_{1,k+1}+\beta_{4,k+1}c_{+,4,k+1}g_{3,4,k+1}\Psi_{2,k+1}+a_{4,k+1}\Psi_{3,k+1}+d_{4,k+1}\Psi_{4,k+1}+b_{4,k+1}\Psi_{5,k+1}+\beta_{4,k+1}c_{-,4,k+1}\sum_{m=2}^{N-4}g_{m+1,4,k+1}\Psi_{4+m,k+1}=$$

$$-\beta_{4,k}c_{+,4,k}g_{4,4,k}\Psi_{1,k}-\beta_{4,k}c_{+,4,k}g_{3,4,k}\Psi_{2,k}-a_{4,k}\Psi_{3,k}-,\beta_{4,k}c_{4,k}\Psi_{4,k}-b_{4,k}\Psi_{5,k}-\beta_{4,k}c_{-,4,k}\sum_{m=2}^{N-4}g_{m+1,4,k+1}\Psi_{4+m,k}$$

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$l=j$

$$\beta_{j,k+1}c_{+,j,k+1}g_{j,j,k+1}\Psi_{1,k+1}+\beta_{J,k+1}c_{+,j},_{k+1}\,g_{j-1,j,k+1}\Psi_{2,k+1}+\ldots+\beta_{J,k+1}c_{+,j,k+1}g_{3,j,k+1}\Psi_{j-2,k+1}$$

$$+a_{j,k+1}\Psi_{j-1,k+1}+d_j\Psi_{j,k+1}+b_{j,k+1}\Psi_{j+1,k+1}+\beta_{j,k+1}c_{-,j,k+1}\sum_{m=2}^{N-j}g_{m+1,j,k+1}\Psi_{j+m,k+1}=$$

$$-\beta_{j,k}c_{+j,k}g_{j,j,k+1}\Psi_{1,k}-\beta_{J,k}c_{+,j,k}g_{j-1}\Psi_{2,k}+\ldots-\beta_{j,k}c_{+,j,k}g_{3,j,k}\Psi_{j-2,k}$$

$$-a_{j,k}\Psi_{j-1,k}-c_{j,k}\Psi_{j,k}-b_{j,k}\Psi_{j+1,k}-\beta_{j,k}c_{-,j,k}\sum_{m=2}^{N-j}g_{m+1,j,k+1}\Psi$$

$l=N-2$

$$\beta_{N-2,k+1}c_{+,N-2,k+1}g_{N-2,N-2,k+1}\Psi_{1,k+1}+\beta_{N-2,k+1}c_{+,N-2,k+1}g_{N-3,N-2,k+1}\Psi_{2,k+1}+\ldots+\beta_{N-2,k+1}c_{+,N-2,k+1}g_{3,N-2,k+1}\Psi_{N-4,k+1}$$

$$+a_{N-2,k+1}\Psi_{N-3,k+1}+d_{N-2,k+1}\Psi_{N-2,k+1}+b_{N-2,k+1}\Psi_{N-1,k+1}=$$

$$-\beta_{N-2,k}c_{+,N-2,k}g_{N-2,N-2,k+1}\Psi_{1,k}-\beta_{N-2,k}c_{+,N-2,K}g_{N-3}\Psi_{2,k}-\ldots-\beta_{N-2,k}c_{+,N-2,k+1}g_{3,N-2,k}\Psi_{N-4,k}$$

$$-a_{N-2,k}\Psi_{N-3,k}-c_{N-2,k}\Psi_{N-2,k}-b_{N-2,k}\Psi_{N-1,k}$$

$l=N-1$

$$\beta_{N-1,k+1}c_{+,N-1,k+1}g_{N-1,N-1,k}\Psi_{1,k+1}+\beta_{N-1,k+1}c_{+,N-1,k}g_{N-2,N-1,k+\&}\Psi_{2,k+1}+\ldots+\beta_{N-1,k+1}c_{+}g_{3,N-1,k+1}\Psi_{N-3,k}+$$

$$a_{N-1,k+1}\Psi_{N-2,k+1}+d_{N-1,k}\Psi_{N-1,k+1}=$$

$$-\beta_{N-1,k}c_{+,N-1,k}g_{N-1,N-1,k}\Psi_{1,k}-\beta_{N-1,k}c_{+,N-1,k}g_{N-2,N-1,k}\Psi_{2,k}-\ldots-\beta_{N-1,k}c_{+}g_{3,N-2,k}\Psi_{N-3,k}-$$

$$a_{N-1,k}\Psi_{N-2,k}-c_{N-1,k}\Psi_{N-1,k}$$

Scheme (3.5) with the boundary condition (3.6) can be written after some simplification in the matrix form as:

$$A^{k+1}\Psi^{k+1} = B^{k}\Psi^{k} \quad 0 \le k \le N-1 \qquad (3.7)$$

$$\Psi^{k+1} = \begin{bmatrix} \Psi_{1,k+1} \\ \bullet \\ \bullet \\ \Psi_{l,k+1} \\ \bullet \\ \bullet \\ \Psi_{N-1,k+1} \end{bmatrix} \qquad \Psi^{k} = \begin{bmatrix} \Psi_{1,k} \\ \bullet \\ \bullet \\ \Psi_{l,k} \\ \bullet \\ \bullet \\ \Psi_{N-1,k} \end{bmatrix}$$

$$A^{k+1}=\begin{bmatrix} d_{1,k+1} & b_{1,k+1} & \beta_{1,k+1}c_{-,1,k+1}g_{3,1,k+1} & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot & \beta_{1,k+1}c_{-,1,k+1}g_{N-1,1,k+1} \\ a_{2,k+1} & d_{2,k+1} & b_{2,k+1} & \beta_{2,k+1}c_{-,2,k+1}g_{3,2,k+1} & \cdot & \cdot\;\cdot\;\cdot & \cdot & \beta_{2,k+1}c_{-,2,k+1}g_{N-2,2,k+1} \\ \beta_{3,k+1}c_{+,3,k+1}g_{3,3,k+1} & a_{3,k+1} & d_{3,k+1} & b_{3,k+1} & \beta_{3,k+1}c_{-,3,k+1}g_{3,3,k+1} & \cdot\;\cdot\;\cdot & \cdot & \beta_{3,k+1}c_{-,3,k+1}g_{N-3,3,k+1} \\ \beta_{4,k+1}c_{+,4,k+1}g_{4,4,k+1} & \cdot & a_{4,k+1} & d_{4,k+1} & b_{4,k+1} & \cdot\;\cdot\;\cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & b_{N-3,k+1} & \cdot \\ \beta_{N-2,k+1}c_{+,N-2,k+1}g_{N-2,N-2,k+1} & \cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & d_{N-2,k+1} & b_{N-2,k+1} \\ \beta_{N-1,k+1}c_{+,N-1,k+1}g_{N-2,N-2,k+1} & \cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & a_{N-1,k+1} & d_{N-1,k+1} \end{bmatrix}$$

$$B^{k}=-\begin{bmatrix} c_{1,k} & b_{1,k} & \beta_{1,k}c_{-,1,k}g_{3,1,k} & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot & \beta_{1,k}c_{-,1,k}g_{N-1,1,k} \\ a_{2,k} & c_{2,k} & b_{2,k} & \beta_{2,k}c_{-,2,k}g_{3,2,k} & \cdot & \cdot\;\cdot\;\cdot & \cdot & \beta_{2,k}c_{-,2,k}g_{N-2,2,k} \\ \beta_{3,k}c_{+,3,k}g_{3,3,k} & a_{3,k} & c_{3,k} & b_{3,k} & \beta_{3,k}c_{-,3,k}g_{3,3,} & \cdot\;\cdot\;\cdot & \cdot & \beta_{3,k}c_{-,3,k}g_{N-3,3,k} \\ \beta_{4,k}c_{+,4,k}g_{4,4,k} & \cdot & a_{4,k} & c_{4,k} & b_{4,k} & \cdot\;\cdot\;\cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & b_{N-3,k} & \cdot \\ \beta_{N-2,k}c_{+,N-2,k}g_{N-2,N-2,k} & \cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & c_{N-2,k} & b_{N-2,} \\ \beta_{N-1,k}c_{+,N-1,k}g_{N-2,N-2,k} & \cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & a_{N-1,k} & c_{N-1,k} \end{bmatrix}$$

4 Numerical simulations

To demonstrate the effectiveness of the method, we present one numerical examples, and, we will compare our numerical results with those obtained using the exact solution

4.1 Example

Consider the space fractional Schrodinger equation with the quantum Riesz-Feller derivative with potential *v*. see [3]

$$\frac{\partial \Psi(x,t)}{\partial t} = -iD_{\theta(x,t)}^{\alpha(x,t)}\Psi(x,t) - iv(x,t)\Psi(x,t) \qquad (4.1)$$

$$0 < x < 2\pi \;;\; 0 < t \le 1 \;;\; 1 < \alpha(x,t) \le 2 \text{ and } \alpha(x,t) = 1.5 + e^{-(xt)^2-1} \;;\theta(t,x) = 0.2$$

$$v(x,t) = \frac{3}{2} + \cos(x - \frac{\theta(x,t)\pi}{2})$$

with a initaial condition :

$$\Psi(x) = \sin x \qquad \Psi(0,t) = \Psi(2\pi,t) = 0$$

where the exact solution is :

$$\Psi(x,t) = \sin x e^{-3i\frac{t}{2}}$$

4.2 Remark

- It's reduce form of Schrodinger equation
- For a discretization we take N=10

4.3 Progamming under Matlab

```
L=input('simlation of x L=')
N=input('discret  N=')
deltax=L/N
T=input('similation of t T=')
deltat=(T/L)*deltax
teta=input('skewness teta=')
C=input ('constant C=')
h=input('planck h=')
k0=1
for l=1:N-1
alp=1.5+0.5*(1/exp(1))
c10(l)=(sin(((alp-teta)*(pi/2)))/sin(alp*pi))
c20(l)=(sin(((alp+teta)*(pi/2)))/sin(alp*pi))
cp1=1;
V0(l)=3/2+2*(cos((deltax*l)-(teta*pi/2)))/sin(deltax*l)

for j=1:N
    g0(j,l)=(1-(1+alp)/j)*cp1;
        cp1=g0(j,l);
end
g1=cp1
a0(l)=(c10(l)*g0(2,l)+c20(l)*g1)/(deltax)^alp
b0(l)=(c10(l)*g1+c20(l)*g0(2,l))/(deltax)^alp
d0(l)=(c10(l)*g0(1,l)+c20(l)*g0(1,l)
-(2*1i*((deltax).^alp)*h)/(deltat*C))/(deltax)^alp
c0(l)=(c10(l)*g0(1,l)+c20(l)*g0(1,l)+2*1i*h*
((deltax).^alp/deltat*C)*(1-1i*deltat*V0(l)/h))/(deltax)^alp
end
```

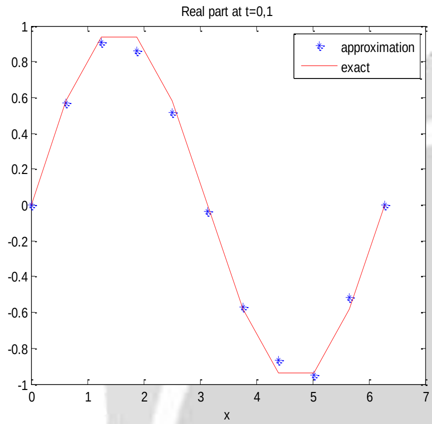
```
B0=zeros(N-1,N-1)
for m=1:N-1
        B0(m,m)=-c0(m)
end

for j=1:N-2
    B0(j,j+1)=-b0(j)
end
for n=3:N-1
    for m=1:(N-1)-(n-1)
    B0(m,m+n-1)=-(c20(m))*g0(n,m)
end
end
for j=2:N-1
    B0(j,j-1)=-a0(j)
end
for n=3:N-2
   B0(N-1,1)=(-c10(N-1)*g0(N-1,N-1))/(deltax)^alp
    B0(n,1)=(-c10(n)*g0(n,n))/(deltax)^alp
 for m=1:(N-1)-n
    B0(n+m,1+m)=(-c10(n+m)*g0(n,n+m))/(deltax)^alp
end

end
for k=1:N
for l=1:N-1
alpha(l,k)=1.5+0.5*(1/exp(+(l*deltax*k*deltat)^2+1))
bet(l,k)=1/(deltax)^alpha(l,k)
c1(l,k)=(sin(((alpha(l,k)-teta)*(pi/2)))/sin(alpha(l,k)*pi))
c2(l,k)=(sin(((alpha(l,k)+teta)*(pi/2)))/sin(alpha(l,k)*pi))
cp1=1;
V(l)=3/2+2*(cos((deltax*l)-(teta*pi/2)))/sin(deltax*l)

for j=1:N
    g(j,l,k)=(1-(1+alpha(l,k)/j))*cp1;
        cp1=g(j,l);
end
g1=1
a(l,k)=bet(l,k)*(c1(l,k)*g(2,l,k)+c2(l,k)*g1)
b(l,k)=bet(l,k)*(c1(l,k)*g1+c2(l,k)*g(2,l,k))
d(l,k)=bet(l,k)*(c1(l,k)*g(1,l,k)+c2(l,k)*g(1,l,k)+
(2*1i*((deltax).^alpha(l,k)*h)/(deltat*C)))
c(l,k)=bet(l,k)*(c1(l,k)*g(1,l,k)+c2(l,k)*g(1,l,k)+
2*1i*h*((deltax).^alpha(l,k)/deltat*C)*(1-1i*deltat*V(l)/h))
end
end
```

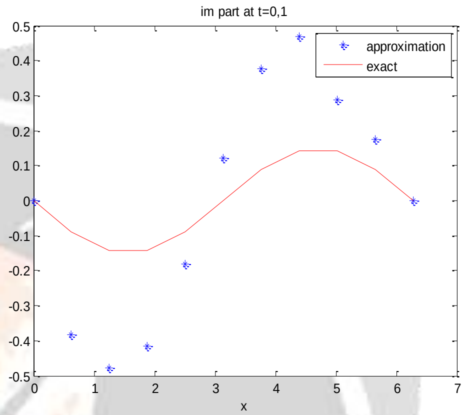
```
for k=1:N
B(:,:,k)=zeros(N-1,N-1)
for m=1:N-1
        B(m,m,k)=-c(m,k)
 end

 for j=1:N-2
    B(j,j+1,k)=-b(j,k)
end
for n=3:N-1
 for m=1:(N-1)-(n-1)
    B(m,m+n-1,k)=-bet(m,k)*((c2(m,k))*g(n,m,k))
end
end
for j=2:N-1
    B(j,j-1,k)=-a(j,k)
end
for n=3:N-2
   B(N-1,1,k)=-bet(N-1,k)*(c1(N-1,k)*g(N-1,N-1,k))
    B(n,1,k)=-bet(n,k)*(c1(n,k)*g(n,n,k))
for m=1:(N-1)-n
    B(n+m,1+m,k)=-bet(n+m,k)*(c1(n+m,k)*g(n,n+m,k))
end

end
end
for k=1:N
 A(:,:,k)=zeros(N-1,N-1)

 for m=1:N-1
        A(m,m,k)=d(m,k)
 end

 for j=1:N-2
    A(j,j+1,k)=b(l,k)
end
for n=3:N-1
 for l=1:(N-1)-(n-1)
    A(l,l+n-1,k)=bet(l,k)*((c2(l,k))*g(n,m,k))
end
end
for l=2:N-1
    A(j,j-1,k)=a(l,k)
end
```

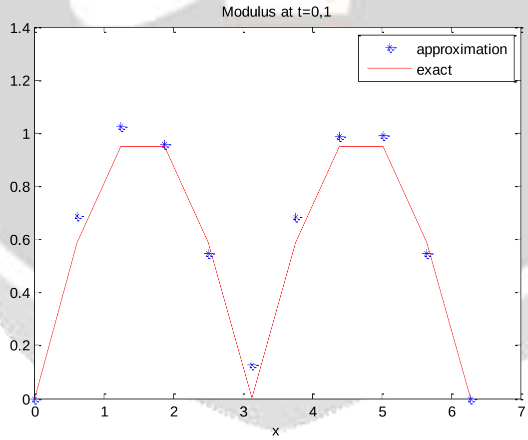
```
for n=3:N-2
    A(N-1,1,k)=bet(1,k)*(c1(1,k)*g(N-1,N-1,k))
    A(n,1,k)=bet(1,k)*(c1(1,k)*g(n,n,k))
for m=1:(N-1)-n
    A(n+m,1+m,k)=bet(1,k)*(c1(1,k)*g(n,n+m,k))
end
end
end

 for l=1:N-1
      U(l)=sin(deltax*l)
      v0=U'
  end

v=zeros(N-1,N-1)
B1(:,:,1)=B0
v(:,1)=A(:,:,1)\B1(:,:,1)*v0
for s=1:N-1

  v(:,s+1)=A(:,:,s+1)\B(:,:,s)*v(:,s)
end
for k=1:N-1


for l=2:N
W(1,k)=0
W(l,k)=v(l-1,k)
W(N+1,k)=0
end
for l=1:N+1
   W1(l,k)=real(W(l,k))
   W2(l,k)=imag(W(l,k))
   S1(l,k)=abs(W(l,k))

end

end
x=[0:deltax:L]
y=W1(:,k0)'
plot(x,y,'*')
hold on
y1=sin(x)*cos((3*(k0*deltat)/2))
plot(x,y1,'r')
figure(2)
r=W2(:,k0)'
plot(x,r,'*')
hold on
```

```
r1=sin(x)*sin((-3*(k0*deltat)/2))
plot(x,r1,'r')
figure(3)
p=S1(:,k0)
plot(x,p,'*')
hold on
p1=abs(sin(x))
plot(x,p1,'r')
```

4.4 Results of the programming/ Graphicals representations







5 Conclusion

In this paper, the shifted Grünwald-letnikov is used to solve the problem . An axample is given. There may be other questions to answer that we may see next time All results in this paper are obtained using MATLAB (2013a).

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