Path Planning in a Plane: A Fluid Analogue

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ABSTRACT

Robotics is the branch of mechatronics which deals with the study of robot. The path of a robot is of tremendous interest of development of technology. Here we have discussed the path of a robot in a plane. The path is an analogue of stream line of uniform flow of an ideal fluid in a plane. The linear transformation gives the complex velocity potential for the flow and its imaginary part gives stream line (Path).

1. Introduction

Terms one-dimensional, two-dimensional and three-dimensional flows, in fluid mechanics denote the number of Cartesian coordinates needed to describe a fluid flow. In general, it appears that any physical flow is three-dimensional. But these flows are problematic to study and needs to reduce the complexity. The simplification can be attained by neglecting any one of the directions that can reduce three-dimensional flow to two-dimensional flow. Fluid flow is called as two-dimensional when the fluid moves in such a way that at any given instant, the flow pattern in a plane is exactly same as that of in all other parallel planes of the fluid. Functions of one complex variable [1-4] are handy to study the two-dimensional flows or planar flows. Thus, fluid parameters in two variables are required.

The velocity field $F(x, y)$, for an ideal fluid [5,6] in the plane is represented by

$$F(x, y) = \nabla\phi(x, y)$$

where $\phi(x, y)$ is a real-valued function which satisfies the Laplace equation [7]

$$\nabla^2\phi(x, y) = 0$$

that is $\phi(x, y)$ harmonic, its harmonic conjugate $\psi(x, y)$ exists. The function $\phi(x, y)$ and $\psi(x, y)$ are called as velocity potential and stream function and the curves

$$\phi(x, y) = constant \text{ and } \psi(x, y) = constant$$

are equipotential curves and stream lines. The analytic function

$$w = \Omega\,(\mathrm{z}) = \phi(x, y) + \mathrm{i}\,\psi(x, y)$$

is called the complex velocity potential of the flow.

$\left|\frac{dw}{dz}\right|$ gives the speed of the flow.

2. Preliminaries

2.1 Conformal Mapping

Let $\mathrm{w = f(z)}$ be a complex mapping defined in a domain D and let $\mathrm{z_0}$ be a point in D. Then we say that $\mathrm{w = f(z)}$ is **conformal** at $\mathrm{z_0}$ if for every pair of smooth oriented curves $\mathrm{C_1}$ and $\mathrm{C_2}$ in D intersecting at $\mathrm{z_0}$ the angle between $\mathrm{C_1}$ and $\mathrm{C_2}$ at $\mathrm{z_0}$ is equal to the angle between the image curves $\mathrm{C'_1}$ and $\mathrm{C'_2}$ at $\mathrm{f(z_0)}$ in both magnitude and sense. [8]

2.2 Theorem

If f is an analytic function in a domain D containing z_0, and if $f'(z_0) \neq 0$, then $w = f(z)$ is a conformal mapping at z_0.[8]

2.3 Theorem

If $w = \Omega\,(z) = \phi(x, y) + i\,\psi(x, y)$ is a one-to-one conformal mapping of the domain D in the z-plane onto a domain D' in the w-

2.4 Theorem

Suppose $w = \Omega\,(z)$ is a mapping from the domain D in the z-plane to a domain D′ in the w-plane. Let the boundary C of D is mapped onto a horizontal line in the w-plane, then $\overline{\Omega}'(z)$ is a complex representation of uniform flow of an ideal fluid in D if and only if $w = \Omega\,(z)$ is a linear map. Further, equipotential curves and stream lines are straight lines perpendicular to each other [9].

2.5 Theorem

Let $\overline{\Omega}'(z)$ be a complex representation of a flow of an ideal fluid in the domain D. A mapping $w = \Omega\,(z)$ is linear if and only if D an infinite plate in the z-plane.[9].

3 Some Governing Equations [10-16]

3.1 Equation of Continuity

The principle of conservation of matter, in a fluid region, say in the absence of inlets and outlets the amount of fluid remains same. This principle is termed as equation of continuity.

The mathematical form of equation of continuity is

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho\overline{q}) = 0$$

Where $\rho = \rho(x, y, z, t)$ represents the fluid density at any point P (x, y, z) in cartesian, at any instant t.

If the pattern of flow is independent of the time at any instant of time t and location P (x, y, z) then $\frac{\partial\rho}{\partial t} = 0$.

Therefore, equation of continuity becomes

$$\nabla.(\rho\overline{q}) = 0$$

For an incompressible, homogenous fluid, the density is constant in the entire fluid.

Above equation becomes

$$\nabla.\overline{q} = 0$$

3.2 Euler's Equation of motion

At time t if $\overline{F}$ is force per unit mass with fluid density ρ and pressure p of moving fluid with velocity $\overline{q}$ then the Euler's equation is

$$\frac{d\rho}{dt} = \overline{F} - \frac{1}{p}\nabla \mathrm{p}$$

3.3 Bernoulli's Equation

For non-viscos fluid relation between velocity and pressure is the Bernoulli's equation, first developed by Euler

$$p + \frac{1}{2}\rho\overline{q^2} = 0$$

4. Discussion

Fluid flow is mainly governed by the equations in § 3 however the fluid flow as complex velocity potential is given in theorem 2.3, using same technique uniform flow is obtained in theorem 2.4. by theorem 2.5 $\overline{\Omega}'(z)$ gives the motion of fluid in plane in complex form if $w = \Omega(z)$ is linear mapping. So, we determine here few paths, using linear transformation, of robot moving in a plane.

4.1 Path in a Horizontal Plane

Let D be an upper half plane, the domain, $y > 0$ in z-plane

Consider the linear transformation

$$w = \Omega(z) = z + 2$$

$$u(x, y) = x + 2\, ;\, v(x, y) = y$$

The stream lines of the flow, i.e. the family of paths are

$$v(x, y) = constant \text{ or } y = constant.$$

Therefore $\overline{\Omega}'(z) = 1$ is a complex representation of motion of robot.

Therefore, speed is

$$\left|\frac{dw}{dz}\right| = |\Omega'(z)| = \left|\overline{\Omega}'(z)\right| = 1$$

Obviously div $\overline{\Omega}'(z) = 0$ and curl $\overline{\Omega}'(z) = 0$.

Thus, $w = \Omega(z) = z + 2$ gives possible motion.

4.2 Path in an inclined Plane

. Consider the linear transformation

$$w = \Omega(z) = z\, e^{\frac{-i\pi}{4}},$$

Defined in a plane.

The paths of robot (stream lines of the flow) are

$$v(x, y) = constant \text{ or } y - x = constant.$$

Therefore $\overline{\Omega}'(z) = e^{\frac{-i\pi}{4}}$ is a complex representation of possible motipn

Therefore, speed

$$\left|\frac{dw}{dz}\right| = |\Omega'(z)| = \left|\overline{\Omega}'(z)\right| = 1$$

5. Conclusion

Using linear transformation, path can be obtained in either horizontal or inclined plane as a stream line of uniform flow of an ideal fluid in the plane can be directly obtained. However, path on a vertical plane can not be practical importance as the vertical path of robot is not possible in practice.

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