Perturbation Analysis of Density Stratified Fluid Layer in Porous Medium.

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Abstract:

The thermosolutal stability of a viscous, conducting, incompressible and heterogeneous fluid layer confined between two free boundaries is studied in porous medium, when the system being investigated is submerged in a uniform vertical magnetic field. Further the stationary, oscillatory and non-oscillatory modes are investigated.. The study of thermosolutal convection in porous medium in a heterogeneous fluid is of great importance. The mathematical formulation of the stability theory proceeds from the non-linear partial differential equations. If the perturbed solution goes on departing from basic solution, the system is said to be unstable and on the other hand if this perturbed solution approaches to theoretical solution, as the time passes, the system is said to be stable.

Keywords: Thermal diffusivity, monotonic function, temperature ,gradient, thermosolutal convection ,vertical magnetic field.

Let a uniform magnetic field is applied to density stratified fluid layer in a porous medium and let fluid is rotating about vertical axes $(0, 0, >)$ with solute concentration S. Also suppose the fluid is to be taken to be non-homogeneous between two horizontal boundaries and heat is applied from below. Let T' and T'' $(T' > T'')$ denote the values of uniform temperatures at the two boundaries, respectively. Then the equations governing the motion are :

$$\rho\left[\frac{\partial V}{\partial t} + (V.\nabla)V\right] = -\nabla P - \frac{\mu}{K}V + \mu\nabla^2 V + \rho X_i + 2\rho V X \Omega_K + \frac{\mu_e}{4\pi}[(\nabla\times H)\times H] \quad (1)$$

$$\nabla.V = 0 \quad (2)$$

$$\nabla.H = 0 \quad (3)$$

$$\frac{\partial T}{\partial t} + (V.\nabla)T = K_T \nabla^2 T \quad (4)$$

$$\frac{\partial C}{\partial t} + (V.\nabla)C = K_S \nabla^2 C \quad (5)$$

$$\frac{\partial H}{\partial t} + (V.\nabla)H = (H.\nabla)V + \eta\nabla^2 H \quad (6)$$

$$\rho = \rho_0[f(z) + \alpha(T_0 - T_1) + \alpha_1(C_0 - C_1)] \quad (7)$$

where $\rho,\ P,\ \mu_e,\ V\ (u,\ v,\ w),\ \mu,\ K,\ K_T,\ K_S$ and C denote respectively, the density, pressure, magnetic permeability, velocity component of the fluid, viscosity, medium permeability, thermal diffusivity of the fluid, coefficient of solute diffusion and solute concentration. ρ_0 is the density of the fluid at the lower boundary at $z=0$. The whole system under the force of gravity $X_i(0,\ 0,\ -g)$ and the $f(z)$ is a monotonic function of vertical co-ordinate z with $f(0)=1$.

Let the initial state of the system be characterized by the following solution for velocity of the fluid, temperature, solute concentration, density, pressure and magnetic field, respectively, as

$$V=(0,\ 0,\ 0) \tag{8}$$

$$T_1 = T_0 - \beta z,\ \beta = (T_0 - T_1)/d \tag{9}$$

$$C_1 = C_0 - \beta_1 z \tag{10}$$

$$\rho = \rho_0[f(z) + \alpha(T_0 - T_1) + \alpha_1(C_0 - C_1)] \tag{11}$$

$$P_1 = -\int g\rho\, dz \tag{13}$$

$$>_K = (0,\ 0,\ >)$$

$$H = (H,\ 0,\ 0) \tag{14}$$

where β represents the uniform adverse temperature gradient maintained between the plates and β_1 represents the solute concentration decreases upward.

To analyse the stability of the fluid we perturb the basic state of the fluid given by equations (8) to (14). Let perturbed state of the fluid layer be given by

$$\overset{*}{v} = (u,\ v,\ w) \tag{15}$$

$$\overset{*}{T_1} = T_1 + \theta \tag{16}$$

$$\overset{*}{C_1} = C_1 + S \tag{17}$$

$$\overset{*}{\rho} = \rho_0\left[\ f(z) + \frac{\delta p}{\rho_0} + \alpha(T_0 - T_1 - \theta) + \alpha_1(C_0 - C_1 - S)\right] \tag{18}$$

$$\overset{*}{P} = P + \delta P \tag{19}$$

$$\overset{*}{H} = (H + h_x,\ h_y,\ h_z) \tag{20}$$

where $(u,\ v,\ w),\ \theta,\ S,\ \delta\rho,\ \delta P$ and $(h_x,\ h_y,\ h_z)$ are respectively, the perturbation in the velocity of the fluid, temperature, solute concentration, density, pressure and the magnetic field. Substituting these variables in the equations (1) to (7) and taking the perturbation variables to be arbitrary small, the linearized perturbation equations are as:

$$\rho_0\frac{\partial u}{\partial t} - 2\rho_0\Omega v = -\frac{\partial \delta P}{\partial x} - \frac{\mu}{K}u + \mu\nabla^2 u + \frac{\mu_e}{4\pi}H\frac{\partial h_x}{\partial x}$$

$$\rho_0\frac{\partial v}{\partial t} - 2\rho_0\Omega u = -\frac{\partial \delta P}{\partial y} - \frac{\mu}{K}v + \mu\nabla^2 v + \frac{\mu_e}{4\pi}H\frac{\partial h_y}{\partial x}$$

$$\rho_0\frac{\partial w}{\partial t} = -\frac{\partial \delta P}{\partial z} - \frac{\mu}{K}w + \mu\nabla^2 w + \frac{\mu_e}{4\pi}H\frac{\partial h_z}{\partial x} - g\delta\rho + g\alpha\,\rho_0\theta - g\alpha_1\rho_0 S$$

$$\frac{\partial(\delta P)}{\partial t} = -\rho_0 w\frac{\partial f}{\partial z}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\frac{\partial h_x}{\partial x}+\frac{\partial h_y}{\partial y}+\frac{\partial h_z}{\partial z}=0$$

$$\frac{\partial \theta}{\partial t}-\beta w=K_T\nabla^2\theta \qquad \frac{\partial S}{\partial t}-\beta_1 w=K_S\nabla^2 S$$

$$\frac{\partial h_x}{\partial t}=H\frac{\partial u}{\partial x}+\eta\nabla^2 h_x$$

$$\frac{\partial h_y}{\partial t}=H\frac{\partial v}{\partial x}+\eta\nabla^2 h_y \quad \text{and} \quad \frac{\partial h_z}{\partial t}=H\frac{\partial w}{\partial x}+\eta\nabla^2 h_z$$

Sufficient Conditions for the stability of the System :

In this section, we analyze the nature of perturbation modes, for this we will solve the eigen value problem consisting of following equations

$$[(D^2-a^2)(D^2-a^2-P_1\sigma)(D^2-a^2-B-\sigma)+Qa_x^2(D^2-a^2)$$

$$-\frac{R_2a^2}{P_r\sigma}(D^2-a^2-P_1\sigma)]w=\left(\frac{2\Omega d^3}{\nu}\right)Dz+Ra^2\left(\frac{K_T}{\beta d^2}\right)(D^2-a^2-P_1\sigma)\theta$$

$$-R_1a^2\left(\frac{K_T}{\beta_1 d^2}\right)(D^2-a^2-P_1\sigma)S$$

$$[(D^2-a^2-P_1\sigma)(D^2-a^2-B\sigma)+Qa_x^2]z=-\left(\frac{2\Omega d}{\nu}\right)(D^2-a^2-P_1\sigma)Dw$$

$$(D^2-a^2-P_r\sigma)\theta=-\left(\frac{\beta d^2}{K_T}\right)w$$

$$(\tau(D^2-a^2)-P_r\sigma)S=-\left(\frac{\beta_1 d^2}{K_T}\right)w$$ together with the boundary conditions

$$\left.\begin{aligned} w=D^2w=Dz=0\\ \theta=S=L=h_z=0\end{aligned}\right\}\text{ at } z=0 \text{ and } z=d$$ and take the solution of the form $w=A\sin n\pi z$,

A is constant. We take the smallest value to n, that is, $n=1$ and so take the solution as

$$w=A\sin\pi z$$

we get,

$$[(\pi^2+a^2)(\pi^2+a^2+P_1\sigma)(\pi^2+a^2+B+\sigma)+Qa_x^2(\pi^2+a^2)$$

$$-\frac{R_2a^2}{P_r\sigma}(\pi^2+a^2+P_1\sigma)]w=-\left(\frac{2\Omega d^3}{\nu}\right)Dz-\left(\frac{Ra^2K_T}{\beta d^2}\right)(D^2-a^2-P_1\sigma)\theta$$

$$+R_1a^2\left(\frac{K_T}{\beta_1 d^2}\right)(D^2-a^2-P_1\sigma)S$$

we find the particular solution for $Dz,\ \theta$ and S, they are

$$Dz = -\left(\frac{2\Omega d}{\nu}\right)\frac{\pi^2(\pi^2 + a^2 + P_1\sigma)}{[(\pi^2 + a^2 + P_1\sigma)(\pi^2 + a^2 + B + \sigma) + Qa_x^2]}w$$

$\theta = \left(\frac{\beta d^2}{K_T}\right)\frac{1}{(\pi^2 + a^2 + P_r\sigma)}w$, and

$$S = \left(\frac{\beta_1 d^2}{K_T}\right)\frac{1}{[\tau(\pi^2 + a^2) + P_r\sigma]}w$$

Now eliminating Dz, θ and S from above equations, we have

$$A_0(A_0 + P_1\sigma)(A_1 + \sigma) + Qa_x^2 A_0 - \frac{R_2 a^2}{P_r\sigma}(A_0 + P_1\sigma)$$

$$= \frac{T\pi^2(A_0 + P_1\sigma)}{[(A_0 + P_1\sigma)(A_1 + \sigma) + Qa_x^2]} + \frac{Ra^2(A_0 + P_1\sigma)}{(A_0 + P_r\sigma)} - \frac{R_1 a^2(A_0 + P_1\sigma)}{(\tau A_0 + P_r\sigma)}$$

where $A_0 = \pi^2 + a^2$ and $A_1 = A_0 + B$

above equation can be written in the following form also,

$$\frac{Ra^2}{(A_0 + P_r\sigma)} = A_0(A_1 + \sigma) + \frac{Qa_x^2 A_0}{(A_0 + P_1\sigma)} - \frac{R_2 a^2}{P_r\sigma} + \frac{R_1 a^2}{\tau A_0 + P_r\sigma}$$

$$-\frac{T\pi^2}{[(A_0 + P_1\sigma)(A_1 + \sigma) + Qa_x^2]}$$

Since σ is the complex growth rate of the perturbations and we can express $\sigma = \sigma_r + i\sigma_i$, where σ_r and σ_i real and σ_i represents the oscillatory character of the perturbations. Substituting the value of σ in equation and taking the real part of the equation, we have for non-oscillatory modes $(\sigma_i = 0)$.

$$\sigma^2\left[\frac{T\pi^2 P_1}{(A_2 + A_3\sigma_r + P_1\sigma_r^2)^2}\right] + \sigma_r\left\{\left[a^2\frac{R}{|A_0 + P_r\sigma|^2} - \frac{R_1}{|\tau A_0 + P_r\sigma|^2}\right]P_r\right.$$

$$\left.+\left[\frac{R_2 a^2}{|\sigma|^2 P_r} - \left(1 + \frac{Qa_x^2 P_1}{|A_0 + P_1\sigma|^2}A_0\right)\right] + \frac{T\pi^2 A_3}{(A_2 + A_3\sigma_r + P_1\sigma_r^2)}\right\}$$

$$+\left\{a^2\left(\frac{R}{|A_0 + P_r\sigma|^2} - \frac{R_1\tau}{|\tau A_0 + P_r\sigma|^2}A_0\right) + \left[\frac{T\pi^2 A_2}{(A_2 + A_3\sigma_r + P_1\sigma_r^2)^2}\right.\right.$$

$$\left.\left.-\left(A_0 A_1 + \frac{Qa_x^2}{|A_0 + P_1\sigma|^2}\right)\right]\right\} = 0$$

where $A_2 = A_0 A_1 + Qa_x^2$ and $A_3 = A_0 + P_1 A_1$

It is clear from the above equation that if

$$R > \max[R_1(L, \tau L)],\ L = \frac{|\ A_0 + P_r\sigma\ |^2}{|\ \tau A_0 + P_r\sigma\ |^2}$$

$$R_2 > \frac{|\ \sigma\ |^2 P_r A_0}{a^2}\left[1 + \frac{Qa_x^2 P_1}{|\ A_0 + P_1\sigma\ |^2}\right] \text{ and}$$

$$T > \frac{(A_2 + A_2\sigma_r + P_1\sigma_r^2)^2}{A_2\pi^2}\left[A_0 A_1 + \frac{Qa_x^2 A_0^2}{|\ A_0 + P_1\sigma\ |^2}\right]$$

then all terms of equation are positive, hence all the roots of the equation are negative. Therefore, equation represents the sufficient conditions for the stability of the system.

References:

1. Agrawal, S.C. and Agrawal, G.S.	1969	Hydromagnetic stability of heterogeneous shear flow. J. Phys. Soc. Japan. 27, 218.
2. Banerjee, M.B.	1970	Thermal Instability of non-homogeneous fluids; A Mathematical theory I. Ind. J. Pure & Appl. math. 1,257.
3. L. A. Romero	2013	Perturbation theory for polynomials, Lecture Notes, University of New Mexico (2013)