# SOLVING DIFFERENTIAL EQUATIONS BY POLYNOMIAL INTEGRAL TRANSFORM

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## ABSTRACT

*In this paper,we discussed about polynomial integral transform for solving differential equations and also we use laplace transform polynomial integral transform solving differential equations with a little effort.Here the integral transform entails the function as its kernel. The Cauchy-Euler method transforms a linear differential equation into an algebraic equation and also we have discussed about the characterization of polynomial integral transform and its properties by using the differential equations.*

## KEYWORDS

*Polynomial integral transform,Polynomial function, Kernel,Differential equations*

### 1.0 Introduction

There are many approaches to search for solution to differential equation with variable coefficients. The Cauchy-Euler method transforms a linear differential equation into an algebraic equation with the use of appropriate substitution technique. In addition these classical methods for search of solutions to the differential equations are tedious and cumber-some as one has to look for the appropriate substitution expression. Thus, there is no single substitution expression for a single type of differential equation.

Nowadays integral transform method is the concern of mathematicians and scientists in general. Since the introduction of the Laplace integral transform, have been proposed for solving differential equations. An alternative integral transform, laplace substitution, for the construction of solutions of the partial differential equation was observed.

We follows the outline of the paper

In section 1, we give the introduction to integral transform method. In this section we discuss the integral transform methods for solving differential equations.

In section 1,2, we present the definition and also give the proof of the polynomial integral transform. Using the polynomial integral transform, we show that the solution of the differential equation converges for $x \in [1, \infty)$.

In section 3, we discuss about the properties of the polynomial integral transform

In section 4, we apply the polynomial integral transform to derivatives, some ordinary differential equation and partial differential equation.

In section 5, it contains the conclusion of the paper.

### 1.1Definition

An **Integral Transform** is any transform $T$ of the following form

$$(Tg)(u) = \int_{k_1}^{k_2} k(t,u)g(t)dt$$

**1.2 Definition**

An equation involving derivatives or differentials of one or more dependent variables with respect to one or more independent variables is called **differential equation.**

**1.3 Definition**

A **polynomial** of degree *n* is a function of the form

$$g(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

**1.4 Definition**

Let $f(x)$ be a function defined for $x \geq 1$.Then the integral

$$B(g(x)) = G(s) = \int_1^\infty g(Inx).x^{-s-1}dx,$$

is the **polynomial integral transform** of $g(x)$ for $x \in [1,\infty)$.

**1.5 Definition**

For any function $g: A \rightarrow B$ (where A and B are any sets), **the kernel**(also called the null space ) is defined by

$$\ker(g) = \{x: x \in Asuchthatg(x) = 0$$

**1.6 Definition**

A linear differential equation of the form

$$a_0 x^n \left(\frac{d^n y}{dx^n}\right) + a_1 x^{n-1}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) + \cdots + a_{n-1}\left(\frac{dy}{dx}\right) + a_n y = X$$

i.e) $(a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + \cdots + a_{n-1} xD + a_n)y = X$

Where $a_0, a_1, \ldots a_n$ are constants and *X* is either a constant or a function of *x* only is called a **homogeneous linear differential equation**. Note that the index of $x$ and the order of derivative is same in each term of such equations. These are also known as **Cauchy-Euler** equations.

## 2.0 The polynomial integral transform

**2.1 Theorem (A Polynomial integral transform)**

Let$g(x)$be a function defined for $x \geq 1$.Then the integral

$$B(g(x)) = G(s) = \int_1^\infty g(Inx).x^{-s-1}dx,$$

is the polynomial integral transform of $g(x)$ for $x \in [1,\infty)$, provided the

integral converges.

**Proof**

We consider the homogeneous Cauchy-Euler equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1 x\frac{dy}{dx} + a_0 y = 0.$$

With the corresponding distinct roots

$$y(x) = c_1 e^{s_1 Inx} + c_2 e^{s_2 Inx} + \cdots + c_n e^{s_n Inx},$$

Where $c_1, c_2, \dots, c_n$are constants.Also , consider a constant linear differential equation

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = 0.$$

With a solution $y(x) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + \cdots + c_n e^{s_n t},$

We can see that equation (3) has an integral transform $B(g(t)) = G(s) = \int_0^\infty g(t) e^{-st} dt$

Again , we see from equation (2) and (4) that

$$t = Inx$$

Substituting equation (6) into equation (5), we obtain

$$B(g(x)) = G(s) = \int_1^\infty g(Inx).\frac{1}{x} e^{-sInx} dx \ = \int_1^\infty f(Inx).x^{-(s+1)}\, dx$$

is the polynomial integral transform of $g(x)$ for $x \in [1, \infty)$, provided the integral converges.

Hence the proof

### 2.1.1 The convergence of the polynomial integral transform

In this section we show that the polynomial integral transform converges for variable defined in $[1, \infty)$.

By Taylor series expansion, we obtain

$$e^{In\, x^{-s-1}} = 1 + In\, x^{-s-1} + \frac{(In\, x^{-(s+1)})^2}{2!} + \frac{In\, x^{-(s+1)})^3}{3!} + \cdots + \sum_{n=0}^{\infty} \frac{(In\, x^{-(s+1)})^n}{n!} + \cdots.$$

$$e^{In\, x^{-s-1}} = \sum_{n=0}^{\infty} \frac{(In\, x^{-(s+1)})^n}{n!}$$

By the D'Lambert Ratio test, we obtain

$$\lim_{n\to\infty} \left| (\sum_{n\to 0}^{\infty} \frac{(In\, x^{-(s+1)})^{n+1}}{(n+1)!} \div \sum_{n=0}^{\infty} \frac{(In\, x^{-(s+1)})^n}{n!}) \right|$$

$$\Rightarrow \ 0.In\, x^{-(s+1)} \ \Rightarrow 0.$$

Then $B\big(g(x)\big) = \sup_{1\le x<\infty} \int_1^\infty |g(In\, x).x^{-s-1}|\, dx$

$$B(g(x)) \le \sup_{1\le x<\infty} \int_1^\infty |g(In\, x)|\, |x^{-s-1}| dx$$

$$B(g(x)) \le M \int_1^\infty |g(In\, x)|\, dx$$

Where $M > 0$.

It implies that the polynomial integral transform converges uniformly for a given $s$.

The function $g(x)$ must be piecewise continuous.

Thus, $g(x)$has at most a finite number of discontinuities on any interval $1 \leq x \leq A$,and the limit of $g(x)$ exist at every point of discontinuity.

### 2.1.2 Existence of the Polynomial Integral Transform

In this section, we show that the Polynomial Integral Transform exists for $x \in [1, \infty)$.

To see this, we state the existence theorem for the Polynomial Integral Transform.

**2.2 Theorem**

Let $g(x)$ be a piecewise continuous function on $[1, \infty)$ and of exponential

order, then the polynomial integral transform exists.

**Proof**

By the definition of polynomial integral transform, we obtain

$$I = \int_1^{\infty} g(x).\, x^{-(s+1)} dx$$

$$= \int_1^{A} g(x).\, x^{-(s+1)} dx + \int_A^{\infty} g(x).\, x^{-(s+1)}\, dx$$

$$I = I_1 + I_2$$

Where$I_1 = \int_1^A g(x).\, x^{-(s+1)}\, dx$

And$I_2 = \int_A^{\infty} g(x).\, x^{-(s+1)} dx$

The integral $I_1$ exists.Since $g(x)$ is piecewise continuous.

Taking $I_2 = \int_A^{\infty} g(x).\, x^{-(s+1)} dx$

$$I_2 = \int_A^{\infty} g(x).\, x^{-(s+1)}\, dx \qquad \leq M \int_A^{\infty} e^{ax}.\, x^{-(s+1)} dx$$

By the Taylor series expansion, we obtain

$$e^{ax} \approx \sum_{n=0}^{\infty} \frac{\propto^n\, x^n}{n!}$$

Substituting the expression for $e^{ax}$in the above equation

We obtain$I_2 \approx M \sum_{n=0}^{\infty} \frac{\propto^n}{n!} \int_A^{\infty} x^{-(s+1-n)}\, dx$

$$I_2 \approx M \sum_{n=0}^{\infty} \frac{\propto^n}{n!} \int_1^{\infty} x^{-(s+1-n)} dx$$

$$I_2 = M \sum_{n=0}^{\infty} \frac{M \propto^n}{n!\,(s-n)}, s > n$$

$$\int_{1}^{\infty} g(x).\, x^{-(s+1)} dx = M \sum_{n=0}^{\infty} \frac{M \propto^{n}}{n!\,(s-n)}, s > n$$

Hence theproof

## 3.0Properties of the Polynomial Integral Transform

In this section, we give the properties of the Polynomial Integral Transform

**3.1 Theorem**

The Polynomial Integral Transform is a linear operator.

**Proof**

Suppose that $f(x)$ and $g(x)$ are functions and $\propto_1$ and $\propto_2$ are real constants.

$$B\big(\propto_1 f(x) + \propto_2 g(x)\big) = \int_{1}^{\infty} (\propto_1 f(In\, x) + \propto_2 g(In\, x)).\, x^{-s-1} dx$$

$$B\big(\propto_1 f(x) + \propto_2 g(x)\big) = \propto_1 \int_{1}^{\infty} f(In\, x).\, x^{-s-1}\, dx + \propto_2 \int_{1}^{\infty} g(In\, x).\, x^{-s-1}\, dx$$

$$B(\propto_1 f(x) + \propto_2 g(x)) = \propto_1 B\big(f(x)\big) + \propto_2 B(g(x))$$

Hence the proof

**3.2 Theorem**

The Inverse Polynomial Integral Transform is also a linear operator

**Proof**

$$B(\propto_1 f(x) + \propto_2 g(x) = \propto_1 B\big(f(x)\big) + \propto_2 B(g(x))$$

Taking both sides the inverse integral transform to the above equation,

We obtain

$$\propto_1 f(x) + \propto_2 g(x) = B^{-1}(\propto_1 B(f(x)) + \propto_2 B(g(x)))$$

$$\propto_1 f(x) + \propto_2 g(x) = B^{-1}(\propto_1 B(f(x))) + B^{-1}(\propto_2 B(g(x)))$$

$$\propto_1 f(x) + \propto_2 g(x) = \propto_1 B^{-1}(B\big(f(x)\big)) + \propto_2 B^{-1}(B\big(g(x)\big))$$

$$\propto_1 f(x) + \propto_2 g(x) = \propto_1 B^{-1}\big(F(s)\big) + \propto_2 B^{-1}(G(s))$$

$$\propto_1 f(x) + \propto_2 g(x) = B^{-1}(\propto_1 \big(F(s)\big) + \propto_2 (G(s)))$$

Where $B(f(x)) = F(s)$ and $B(g(x)) = G(s)$,respectively.

Hence the proof

**3.3 Theorem**

**(First shifting theorem)**

If $B\big(g(x)\big) = B(s)$,then $B\big(e^{ax}g(x)\big) = B(s-a)$, for $s > 1$.

**Proof**

Let $B\big(e^{ax}g(x)\big) = \int_1^\infty e^{aIn\,x} g(In\,x).\,x^{-s-1}\,dx$

$$B\big(e^{ax}g(x)\big) = \int_1^\infty x^a g(In\,x).\,x^{-s-1}dx$$

$$B\big(e^{ax}g(x)\big) = \int_1^\infty g(In\,x).\,x^{-(s-a+1)}\,dx$$

$$B\big(e^{ax}g(x)\big) = B(s-a)$$

Hence the proof

**3.4 Theorem**

**(Second shifting theorem)**

Let $H_c(x) = \begin{cases} 0 & 0 \le x < c \\ 1 & x \ge c \end{cases}$

be a unit step function. Then $B\big(H_c g(x-c)\big) = G(s-c)$

**Proof**

By applying the polynomial integral transform, we obtain

$$B\big(H_c(x)g(x-c)\big) = \int_1^\infty H_c\,(In\,x) g\big(In\,(x-c)\big).\,x^{-s-1}dx$$

$$B\big(H_c(x)g(x-c)\big) = \lim_{t\to\infty}\int_1^t 1.\,g\big(In(x-c)\big).\,x^{-s-1}dx$$

$$B\big(H_c(x)g(x-c)\big) = \lim_{t\to\infty}\int_1^t g\big(In(x-c)\big).\,x^{-s-1}dx$$

We set $u = x - c$and substituting $u$ into R.H.S of the above equation

we obtain$B\big(H_c(x)g(x-c)\big) = \lim_{t\to\infty}\int_{1-c}^{t-c} g(In\,u).\,(u+c)^{-s-1}\,du$

$$B\big(H_c(x)g(x-c)\big) = \lim_{t\to\infty}\int_1^t g\big(In\,(v-c)\big).\,v^{-s-1}dv$$

$$B\big(H_c(x)g(x-c)\big) = G(s-c)$$

where $v = u + c$.

Hence the proof

**3.5 Theorem**

If $g(x)$ is a piecewise continuous function on $[0,\infty)$, but not of exponential order, then a polynomial integral transform

$$B\big(g(x)\big) \to 0\ as\ s \to \infty.$$

**Proof**

Let $|B(g(x))| = \left|\int_1^{\infty} g(In\, x)x^{-s-1}dx\right|$

$$|B(g(x))| \leq \int_1^{\infty} |g(In\, x)x^{-s-1}|\, dx$$

$$|B(g(x))| = \int_1^{\infty} g(In\, x)|x^{-s-1}|\, dx$$

But we observe that $\left|x^{-(s+1)}\right| \to 0\ as\ s \to \infty$

It follows that $B\big(g(x)\big) \to 0\ as\ s \to \infty$.

Hence the proof.

## 4.0 The Polynomial Integral Transform of Derivatives

In this section , we give the polynomial integral transform of derivatives of the function $g(x)$ with respect to $x$.

### 4.1 Theorem

If $g, g^1, \ldots, g^{n-1}$ are continuous on $[1, \infty)$ and if $f^n(x)$ is piecewise continuous on $[1, \infty)$, then

$$B\left(g^{(n)}(x)\right) = s^n G(s) - s^{n-1}g(0) - s^{n-2}g'(0) - \cdots - g^{(n-1)}(0)$$

Where $G(s) = B\big(g(x)\big)$.

**Proof**

Let $B\big(g'(x)\big) = \int_1^{\infty} g'\,(In\, x).\, x^{-s-1}dx$

Using integration by parts, we obtain

$$L\big(g'(x)\big) = \lim_{t\to\infty}[g(In\, x)x^{-s}]_1^t + \lim_{t\to\infty}\int_1^t g(In\, x)\frac{1}{x}x^{-s}dx$$

$$L\big(g'(x)\big) = sG(s) - g(0)$$

Proceeding a similar as above , we obtain

$$B\big(g''(x)\big) = \int_1^{\infty} g''(In\, x).\, x^{-s-1}dx \Rightarrow B\big(g''(x)\big) = -g'(0) + sL(g'(x))$$

Substituting the expression of $g'(x)$ into the above equation , we obtain

$$B\big(g''(x)\big) = s^2G(s) - sg(0) - g'(0)$$

By induction, we obtain

$$B\left(g^{(n)}(x)\right) = s^nG(s) - s^{n-1}g(0) - s^{n-2}g'(0) - \cdots - g^{n-1}(0)$$

Where $G(s) = B\big(g(x)\big)$.

Hence the proof

## Conclusion

We observed that the polynomial integral transform solves differential equation with a few computations as well as time. Unlike the laplace integral transform ,the polynomial integral integral transform involves a polynomial function as its kernel, which is easier and transforms complicated functions into algebraic equations. The solution of the differential equation is then obtained from the algebraic equation. Also, using the polynomial integral transform, the convergence of the solution of the differential equation is faster as compared with the Laplace integral transform and others. We observed that the Polynomial Integral Transform is defined on the interval $[1, \infty)$.

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