SOME PROBLEMS ABOUT ARITHMETIC AVERAGE OF ODER ARBITRARY AND GEOMETRIC AVERAGE

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ABSTRACT

In mathematics, functional equations are very difficult and arise all areas of mathematics, even more, science, engineering, and social sciences. They appear at all levels of mathematics. The theory of functional equations were born very early. Many authors studied functional equations. In this artical, we study some problems about arithmetic average of oder arbitrary and geometric average.

Keyword: *Functional equalities, arithmetic average of oder arbitrary, geometric average.*

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1. PRELIMINARIES

In this artical, we would like to look at some expressions

$$\frac{x+y}{2};\, x, y \in \mathbb{R};$$

$$\sqrt{\frac{x^2+y^2}{2}};\, x, y \in \mathbb{R}^+;$$

$$\sqrt[m]{\frac{x^m+y^m}{2}};\, m=1,2,...;\, x, y \in \mathbb{R}^+;$$

$$\sqrt{f(x)f(y)}$$

In this paper, we use method of substitution

+) Example, let $x=\alpha$ such that $f(\alpha)$ appears much in the equation.

+) Let $x=\alpha, y=\beta$ interchange to refer $f(\alpha)$ and $f(\beta)$.

+) Let $f(0)=\beta, f(1)=\beta$.

+) If f is surjection, exist $\alpha : f(\alpha)=0$. Choice x, y to destroy $f(g(x,y))$ in the equation. The function has x, we show that it is injective or surjection.

+) To occur $f(x)$.

+) $f(x) = f(y)$ for all x, y ∈ A. Hence $f(x) = const$ for all $x \in A$.

On the other hand, we use math induction and the continuos of function.

2. SOME PROBLEMS ABOUT ARITHMETIC AVERAGE OF ODER ARBITRARY AND GEOMETRIC AVERAGE

In this part, we would like to look at some problems about arithmetic average of oder arbitrary and geometric average.

2.1 Problem 1. Determine all functions $f:\mathbb{R} \to \mathbb{R}$ which are continuous on $\mathbb{R}$ and satisfy the equation

$$f\left(\frac{x+y}{2}\right)=\sqrt{f(x)f(y)},\forall x,y\in\mathbb{R}. \quad (*)$$

Solution. Setting $x=\alpha$ and $y=\alpha$, we get

$$f(\alpha)=\sqrt{\left[f(\alpha)\right]^2}=\left|f(\alpha)\right|\geq 0,\forall \alpha\in\mathbb{R}.$$

Then, we have two cases:

+ Case 1: Exist x_0 such that $f(x_0)=0.$ Then, for all $\alpha\in\mathbb{R}$, we have:

$$f(\alpha)=f\left(\frac{x_0+(2\alpha-x_0)}{2}\right)=\sqrt{f(x_0).f(2\alpha-x_0)}=0,\forall \alpha\in\mathbb{R}.$$

So $f(\alpha)\equiv 0$ is a solution of (*).

+ Case 2: $f(\alpha)>0,\forall\alpha\in\mathbb{R}$. Log base of e of (*), we have

$$\ln f\left(\frac{x+y}{2}\right)=\ln\left[f(x)f(y)\right]^{1/2}=\frac{\ln f(x)+\ln f(y)}{2},\forall x,y\in\mathbb{R},$$

or

$$g\left(\frac{x+y}{2}\right)=\frac{g(x)+g(y)}{2},\forall x,y\in\mathbb{R}.$$

with $g(\alpha)=\ln f(\alpha).$

By Problem 1 [1], we have

$$g(\alpha)=a\alpha+b;\forall a,b\in\mathbb{R}.$$

Hence,

$$\ln f(\alpha)=a\alpha+b.$$

So,

$$f(\alpha)=e^{a\alpha+b}=e^b\left(e^a\right)^\alpha=BA^\alpha.$$

We can check directly $f(x)=BA^x;\forall x\in\mathbb{R};$ with for arbitrary $A,B>0$ satisfies (*).

There for,

$$f(x)=BA^x;\forall x\in\mathbb{R}; \text{ with for arbitrary } A,B>0.$$

2.2 Problem 2. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ which are continuous on the real axis and satisfy the equation

$$f\left(\sqrt{\frac{x^2+y^2}{2}}\right)=\sqrt{f(x)f(y)},\forall x,y\in\mathbb{R}. \quad (**)$$

Solution. By assumption, we have $f(x)\geq 0,\forall x\geq 0.$

If exist x_0 such that $f(x_0)=0.$ Then,

$$f\left(\sqrt{\frac{x_0^2+y^2}{2}}\right)=\sqrt{f(x_0)f(y)}=0,\forall y\in\mathbb{R}^+.$$

Hence, $f\left(x\right)\equiv 0,\forall x\geq\frac{|x_0|}{2}.$

Replacing x_0 by $\frac{|x_0|}{\sqrt{2}}$ and by math induction, we get

$$f\left(\frac{|x_0|}{\left(\sqrt{2}\right)^n}\right),\forall n\in\mathbb{N}.$$

Since $f\left(x\right)$ is continuous at $x=0$, then

$$\lim_{n\to\infty}f\left(\frac{|x_0|}{\left(\sqrt{2}\right)^n}\right)=f\left(0\right)=0.$$

Then,

$$f\left(\sqrt{\frac{x_0^2+0}{2}}\right)=f\left(\frac{|x_0|}{\sqrt{2}}\right)=\sqrt{f\left(x_0\right)f\left(0\right)}=0,\forall x\geq 0.$$

Hence $f\left(x\right)\equiv 0,\forall x\geq 0.$

On the other hand, by (**), we get

$$f\left(\sqrt{\frac{x^2+x^2}{2}}\right)=f\left(|x|\right)=\sqrt{\left[f\left(x\right)\right]^2}=\left|f\left(x\right)\right|,\forall x\geq 0.$$

Hence, $f\left(x\right)\equiv 0,\forall x\in\mathbb{R}$.

Assume that, $f\left(x\right)\neq 0,\forall x\in\mathbb{R}$. If exist x_1 such that $f\left(x_1\right)<0$ and by (**), we have

$$f\left(\sqrt{\frac{x_1^2+y^2}{2}}\right)=\sqrt{f\left(x_1\right)f\left(y\right)}=0,\forall y\in\mathbb{R}.$$

Hence, $f\left(y\right)<0,\forall y\in\mathbb{R}$, cotracdiction. So $f\left(x\right)>0,\forall x\in\mathbb{R}$ and

$$(**)\Leftrightarrow \ln f\left(\sqrt{\frac{x^2+y^2}{2}}\right)=\frac{\ln f\left(x\right)+\ln f\left(y\right)}{2},\forall x,y\in\mathbb{R}.$$

Setting $\ln f\left(x\right)=g\left(x\right)$. Then for, $g\left(x\right)$ is continuous on $\mathbb{R}$ and

$$g\left(\sqrt{\frac{x^2+y^2}{2}}\right)=\frac{g\left(x\right)+g\left(y\right)}{2},\forall x,y\in\mathbb{R}.$$

By Problem 2 $\left[1\right]$, we get

$$g\left(x\right)=ax^2+b,\forall x\in\mathbb{R}.$$

Hence,

$$f\left(x\right)=e^{ax^2+b}\ .$$

We can check directly $f\left(x\right)=\mathrm{e}^{\mathrm{a}x^2+b}, \forall a,b\in\mathbb{R}$ satisfies problem.

Hence,

$$f\left(x\right)=\mathrm{e}^{\mathrm{a}x^2+b}, \forall a,b\in\mathbb{R}\ .$$

2.3 Problem 3. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ which are continuous on $\mathbb{R}$ and satisfy the equation

$$f\left(\sqrt[m]{\frac{x^m+y^m}{2}}\right)=\sqrt{f\left(x\right)f\left(y\right)}, \forall x,y\in\mathbb{R}, m=1,2.... \quad (***)$$

Solution. By assumption, we have $f\left(x\right)\geq 0; \forall x\geq 0.$

If exist x_0 such that $f\left(x_0\right)=0.$ Then,

$$f\left(\sqrt[m]{\frac{x_0^m+y^m}{2}}\right)=\sqrt{f(x_0)f(y)}=0, \forall y\in\mathbb{R}^+. \quad (6)$$

Hence, $f\left(x\right)\equiv 0, \forall x\geq\dfrac{|x_0|}{2}.$

Replacing x_0 by $\dfrac{|x_0|}{\sqrt{2}}$ and by math induction, we get

$$f\left(\frac{|x_0|}{\left(\sqrt{2}\right)^n}\right), \forall n\in\mathbb{N}.$$

Since $f\left(x\right)$ continuous at $x=0$, then

$$\lim_{n\to\infty} f\left(\frac{|x_0|}{\left(\sqrt[m]{2}\right)^n}\right)=f\left(0\right)=0.$$

Hence,

$$f\left(\sqrt[m]{\frac{x_0^m+0}{2}}\right)=f\left(\frac{|x_0|}{\sqrt{2}}\right)=\sqrt{f\left(x_0\right)f\left(0\right)}=0, \forall x\geq 0.$$

Hence, $f\left(x\right)\equiv 0, \forall x\geq 0.$

On the other hand, by (***), we get

$$f\left(\sqrt{\frac{x^2+x^2}{2}}\right)=f\left(|x|\right)=\sqrt{\left[f\left(x\right)\right]^2}=\left|f\left(x\right)\right|, \forall x\geq 0.$$

Hence, $f\left(x\right)\equiv 0, \forall x\in\mathbb{R}.$

Assume that, $f\left(x\right)\neq 0, \forall x\in\mathbb{R}$. If exist x_1 such that $f\left(x_1\right)<0$ and by (***), we have

$$f\left(\sqrt[m]{\frac{x_1^m + y^m}{2}}\right) = \sqrt{f(x_1)\, f(y)} = 0, \forall y \in \mathbb{R}.$$

Hence, $f(y) < 0, \forall y \in \mathbb{R}$ cotracdiction. So that $f(x) > 0, \forall x \in \mathbb{R}$, and

$$(***) \Leftrightarrow \ln f\left(\sqrt[m]{\frac{x^m + y^m}{2}}\right) = \frac{\ln\left[f\left(\sqrt[m]{x}\right)\right] + \ln\left[f\left(\sqrt[m]{y}\right)\right]}{2}, \forall x, y \in \mathbb{R}.$$

Let $\ln\left[f\left(\sqrt[m]{x}\right)\right] = g(x)$. Hence, $g(x)$ is continuos on $\mathbb{R}$ and

$$g\left(\sqrt[m]{\frac{x^m + y^m}{2}}\right) = \frac{g(x) + g(y)}{2}, \forall x, y \in \mathbb{R}.$$

By Problem 3 $[1]$, we have $g(x) = ax^m + b, \forall x \in \mathbb{R}$.

There for, $f(x) = e^{ax^m + b}$,

We can check directly $f(x) = e^{ax^m + b}$ satisfies problem.

Hence,

$$f(x) \equiv 0, \text{ or } f(x) = e^{ax^m + b}, \forall a, b \in \mathbb{R}.$$

3. CONCLUSIONS

In this paper, we establish some problems about arithmetic average of oder arbitrary and geometric average. It is very good for teachers and students.

4. REFERENCES

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