Some Operations and Properties of Interval Valued Bipolar Neutrosophic Hesitant Fuzzy Set (IVBNHFS)

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ABSTRACT

Hesitant neutrosophic set is an important extension of hesitant fuzzy set. Bipolar neutrosophic set is an important extension of bipolar fuzzy set. The Interval Valued Bipolar Neutrosophic Hesitant Fuzzy Set (IVBNHFS) is a new generalization of fuzzy set, bipolar fuzzy set, neutrosophic set, hesitant fuzzy set, and bipolar neutrosophic set. The IVBNHFS set is the hybridization of bipolar fuzzy set, hesitant fuzzy set and interval neutrosophic set. Every element of IVBNHFS consists of three independent positive membership functions and three independent negative membership functions. In this paper, we define some operations and we also investigate some of properties for IVBNHFS, such as union, intersection, sum algebraic, ... And we prove some theorems on IVBNHFS. In the end, the main conclusion and future scope of research are summarized.

Keyword: -*Fuzzy set, Hesitant fuzzy set, Neutrosophic set, Hesitant neutrosophic set, Bipolar neutrosophic set, Bipolar fuzzy set, Interval neutrosophic hesitant fuzzy set, Interval bipolar neutrosophic fuzzy set, Interval valued bipolar neutrosophic hesitant fuzzy set.*

1. INTRODUCTION

Neutrosophic set [1] conceived by Smarandache in 1998 is the generalization of fuzzy set [2] given officially by Zadeh in 1965 and intuitionistic fuzzy set [3] presented by Atanassov in 1983. Torra defined hesitant fuzzy set [4] in 2010. Ye suggested the hesitant neutrosophic sets [5] as a generalization of hesitant fuzzy set in 2014. Bipolar fuzzy sets [6] are proved to very efficiency in uncertain problems which can be characterized not only the positive characteristics but also the negative characteristics of a certain problem. Neutrosophic bipolar fuzzy sets are already presented in daily life’s problem [7]. Recently, Al-Quran [8] adopted a hesitant bipolar-valued neutrosophic set (HBVNS) based on the combination of bipolar neutrosophic sets and hesitant fuzzy sets. HBVNS generalized the notions of fuzzy set, intuitionistic fuzzy set, hesitant fuzzy set, single-valued neutrosophic set, single-valued neutrosophic hesitant fuzzy set, bipolar fuzzy set and bipolar neutrosophic set. Further, Al-Quran [8] defined the basic operational laws, union, intersection and complement for hesitant bipolar-valued neutrosophic elements (HBVNEs) and studied its associated properties. Two aggregation operators [8] are developed, based on HBVNS which are the hesitant bipolar-valued neutrosophic weighted averaging (HBVNWA) and the hesitant bipolar-valued neutrosophic weighted geometric (HBVNWG). A decision-making method [8] is developed based on HBVNS and, the HBVNWA and HBVNWG operators. In bipolar neutrosophic sets and hesitant neutrosophic sets, interval valued bipolar neutrosophic hesitant fuzzy set hasn’t been appeared in literature till now. Thus in this research, we present the new interval valued bipolar neutrosophic hesitant fuzzy set (IVBNHFS). The concept of IVBNHFS is introduced by coupling both the interval bipolar neutrosophic fuzzy set [9] and interval neutrosophic hesitant fuzzy set [10]. For convenience, we use authors’ names Hans and Hery Zo Set (2HZS) instead of IVBNHFS.

The rest of the paper is structured as follows. Section 2 presents some basic concept of HBVNS. This section 2 provides the related definitions that will be used in the development of the new IVBNHFS. Section 3 proposes the IVBNHFS and their basic properties in section 4. Section 5 is devoted to present some new theorems related to IVBNHFS operations. Section 6 presents the conclusions and future scope of the research.

2. PRELIMINARIES

2.1 HBVNS

Definition 1: [8]

Let V be a reference set and with a generic element in V denoted by v. A hesitant bipolar-valued neutrosophic set H in V is defined as:

$$H = \{v, \langle h_T^+(v), h_I^+(v), h_F^+(v), h_T^-(v), h_I^-(v), h_F^-(v) \rangle | \; v \in V\} \quad (1)$$

Where $h_T^+(v), h_I^+(v), h_F^+(v) : V \to [0,1]$ and $h_T^-(v), h_I^-(v), h_F^-(v) : V \to [-1,0]$. The positive hesitant bipolar-valued neutrosophic elements, h_T^+, h_I^+ and h_F^+ denotes the possible satisfactory degree of truth, indeterminacy and falsity of an element $v \in V$ corresponding to a HBVNS H respectively while the negative hesitant bipolar-valued neutrosophic elements , h_T^-, h_I^- and h_F^- denote the possible satisfactory degree of truth, indeterminacy and falsity of an element $v \in V$ to the implicit counter property to the set H respectively. In addition, a HBVNS H must satisfy the conditions $0 \le \gamma_T^+, \gamma_I^+, \gamma_F^+ \le 1$, $-1 \le \gamma_T^+, \gamma_I^+, \gamma_F^+ \le 0$, $0 \le max\{\gamma_T^+\} + max\{\gamma_I^+\} + max\,\{\gamma_F^+\} \le 3$, and $-3 \le max\{\gamma_T^-\} + max\{\gamma_I^-\} + max\,\{\gamma_F^-\} \le 1$. In which, $\gamma_T^+ \in h_T^+(v)$, $\gamma_I^+ \in h_I^+(v)$, $\gamma_F^+ \in h_F^+(v)$, $\gamma_T^- \in h_T^-(v)$, $\gamma_I^- \in h_I^-(v)$ and $\gamma_F^- \in h_F^-(v)$ for $v \in V$. For convenience, we use the symbol H to represent all the hesitant bipolar-valued neutrosophic sets and $h = \{h_T^+, h_I^+, h_F^+, h_T^-, h_I^-, h_F^-\}$ for a hesitant bipolar-valued neutrosophic element (HBVNE).

2.2 Properties of HBVNS

Definition 2 [8]

Let $h = \{h_T^+, h_I^+, h_F^+, h_T^-, h_I^-, h_F^-\}$, $h_1 = \{h_{T1}^+, h_{I1}^+, h_{F1}^+, h_{T1}^-, h_{I1}^-, h_{F1}^-\}$ and $h_2 = \{h_{T2}^+, h_{I2}^+, h_{F2}^+, h_{T2}^-, h_{I2}^-, h_{F2}^-\}$ be three HBVNEs. Then,

i) Inclusion

$h_1 \subseteq h_2$ if and only if $\forall\,\gamma_{T1}^+ \le \forall\,\gamma_{T2}^+$, $\forall\,\gamma_{I1}^+ \le \forall\,\gamma_{I2}^+$, $\forall\,\gamma_{F1}^+ \le \forall\,\gamma_{F2}^+$ and $\forall\,\gamma_{T1}^- \ge \forall\,\gamma_{T2}^-$, $\forall\,\gamma_{I1}^- \ge \forall\,\gamma_{I2}^-$, $\forall\,\gamma_{F1}^- \ge \forall\,\gamma_{F2}^-$ for all $v \in V$.

ii) Equality

$h_1 = h_2$ if and only if $\forall\,\gamma_{T1}^+ = \forall\,\gamma_{T2}^+$, $\forall\,\gamma_{I1}^+ = \forall\,\gamma_{I2}^+$, $\forall\,\gamma_{F1}^+ = \forall\,\gamma_{F2}^+$ and $\forall\,\gamma_{T1}^- = \forall\,\gamma_{T2}^-$, $\forall\,\gamma_{I1}^- = \forall\,\gamma_{I2}^-$, $\forall\,\gamma_{F1}^- = \forall\,\gamma_{F2}^-$ for all $v \in V$.

iii) Complement

The complement of h is denoted by h^C and defined as follows:

$$h^C = \left\langle \begin{array}{l} \cup_{\gamma_T^+ \in h_T^+}\{1-\gamma_T^+\}, \cup_{\gamma_I^+ \in h_I^+}\{1-\gamma_I^+\}, \cup_{\gamma_F^+ \in h_F^+}\{1-\gamma_F^+\}, \\ \cup_{\gamma_T^- \in h_T^-}\{-1-\gamma_T^-\}, \cup_{\gamma_F^- \in h_I^-}\{-1-\gamma_I^-\}, \cup_{\gamma_F^- \in h_F^-}\{-1-\gamma_F^-\} \end{array} \right\rangle \quad (2)$$

iv) Union

The union of two HBVNEs is defined by

$$h_1 \cup h_2 = \left\langle \begin{array}{l} \cup_{\substack{\gamma_{T1}^+ \in h_{T1}^+, \\ \gamma_{T2}^+ \in h_{T2}^+}} \max\{\gamma_{T1}^+, \gamma_{T2}^+\}, \cup_{\substack{\gamma_{I1}^+ \in h_{I1}^+, \\ \gamma_{I2}^+ \in h_{I2}^+}} \left\{\frac{\gamma_{I1}^+, \gamma_{I2}^+}{2}\right\}, \cup_{\substack{\gamma_{F1}^+ \in h_{F1}^+, \\ \gamma_{F2}^+ \in h_{F2}^+}} \min\{\gamma_{F1}^+, \gamma_{F2}^+\}, \\ \cup_{\substack{\gamma_{T1}^- \in h_{T1}^-, \\ \gamma_{T2}^- \in h_{T2}^-}} \min\{\gamma_{T1}^-, \gamma_{T2}^-\}, \cup_{\substack{\gamma_{I1}^- \in h_{I1}^-, \\ \gamma_{I2}^- \in h_{I2}^-}} \left\{\frac{\gamma_{I1}^-, \gamma_{I2}^-}{2}\right\}, \cup_{\substack{\gamma_{F1}^- \in h_{F1}^-, \\ \gamma_{F2}^- \in h_{F2}^-}} \max\{\gamma_{F1}^-, \gamma_{F2}^-\} \end{array} \right\rangle \quad (3)$$

v) Intersection

The intersection of two HBVNEs is defined by

$$h_1 \cap h_2 = \left\langle \begin{array}{l} \bigcup_{\substack{\gamma_{T1}^+ \in h_{T1}^+, \\ \gamma_{T2}^+ \in h_{T2}^+}} \min\{\gamma_{T1}^+, \gamma_{T2}^+\}, \bigcup_{\substack{\gamma_{I1}^+ \in h_{I1}^+, \\ \gamma_{I2}^+ \in h_{I2}^+}} \left\{\frac{\gamma_{I1}^+, \gamma_{I2}^+}{2}\right\}, \bigcup_{\substack{\gamma_{F1}^+ \in h_{F1}^+, \\ \gamma_{F2}^+ \in h_{F2}^+}} \max\{\gamma_{F1}^+, \gamma_{F2}^+\}, \\ \bigcup_{\substack{\gamma_{T1}^- \in h_{T1}^-, \\ \gamma_{T2}^- \in h_{T2}^-}} \max\{\gamma_{T1}^-, \gamma_{T2}^-\}, \bigcup_{\substack{\gamma_{I1}^- \in h_{I1}^-, \\ \gamma_{I2}^- \in h_{I2}^-}} \left\{\frac{\gamma_{I1}^-, \gamma_{I2}^-}{2}\right\}, \bigcup_{\substack{\gamma_{F1}^- \in h_{F1}^-, \\ \gamma_{F2}^- \in h_{F2}^-}} \min\{\gamma_{F1}^-, \gamma_{F2}^-\} \end{array} \right\rangle \quad (4)$$

3. 2HZS (IVBNHFS)

Definition 3

Assume X is a non-empty finite set, a 2HZS P on X is defined as:

$$P = \{\langle x, t^+(x), i^+(x), f^+(x), t^-(x), i^-(x), f^-(x)\rangle |\, x \in X\} \quad (5)$$

Where $t^+(x) = \{\gamma^+ | \gamma^+ \in t^+(x)\}, i^+(x) = \{\delta^+ | \delta^+ \in i^+(x)\}$, $f^+(x) = \{\eta^+ | \eta^+ \in f^+(x)\}$ are three positive membership functions expressed by a few close intervals in the real unit interval [0,1] which represent the truth positive membership hesitant degree, indeterminacy positive membership hesitant degree and falsity positive membership hesitant degree and meet the following conditions: lower

$$\gamma^+ = [\gamma_L^+, \gamma_U^+] \in [0,1], \delta^+ = [\delta_L^+, \delta_U^+] \in [0,1], \eta^+ = [\eta_L^+, \eta_U^+] \in [0,1]$$
$$0 \le \gamma_U^+ + \delta_U^+ + \eta_U^+ \le 3$$

And $t^-(x) = \{\gamma^- | \gamma^- \in t^-(x)\}, i^-(x) = \delta^- | \delta^- \in i^-(x)$, $f^-(x) = \{\eta^- | \eta^- \in f^-(x)\}$ are three negative membership functions expressed by a few close intervals in the real unit interval $[-1,0]$ which represent the truth negative membership hesitant degree, indeterminacy negative membership hesitant degree and falsity negative membership hesitant degree and meet the following conditions :

$$\gamma^- = [\gamma_L^-, \gamma_U^-] \in [-1, 0], \delta^- = [\delta_L^-, \delta_U^-] \in [-1, 0], \eta^- = [\eta_L^-, \eta_U^-] \in [-1,0]$$
$$-3 \le \gamma_U^- + \delta_U^- + \eta_U^- \le 0$$

For the rest of the paper, let $A = \langle [t_{LA}^+, t_{UA}^+], [i_{LA}^+, i_{UA}^+], [f_{LA}^+, f_{UA}^+], [t_{LA}^-, t_{UA}^-], [i_{LA}^-, i_{UA}^-], [f_{LA}^-, f_{UA}^-] \rangle$,
$B = \langle [t_{LB}^+, t_{UB}^+], [i_{LB}^+, i_{UB}^+], [f_{LB}^+, f_{UB}^+], [t_{LB}^-, t_{UB}^-], [i_{LB}^-, i_{UB}^-], [f_{LB}^-, f_{UB}^-] \rangle$,
$C = \langle [t_{LC}^+, t_{Uc}^+], [i_{Lc}^+, i_{Uc}^+], [f_{LC}^+, f_{UC}^+], [t_{LC}^-, t_{UC}^-], [i_{LC}^-, i_{UC}^-], [f_{LC}^-, f_{UC}^-] \rangle$, be three 2HZS,
Where
$t_A^+(x) = \{\gamma_A^+ | \gamma_A^+ \in t_A^+(x)\}, i_A^+(x) = \{\delta_A^+ | \delta_A^+ \in i_A^+(x)\}, f_A^+(x) = \{\eta_A^+ | \eta_A^+ \in i_A^+(x)\}$
$t_B^+(x) = \{\gamma_B^+ | \gamma_B^+ \in t_B^+(x)\}, i_B^+(x) = \{\delta_B^+ | \delta_B^+ \in i_B^+(x)\}, f_B^+(x) = \{\eta_B^+ | \eta_B^+ \in i_B^+(x)\}$
$t_C^+(x) = \{\gamma_C^+ | \gamma_C^+ \in t_C^+(x)\}, i_C^+(x) = \{\delta_C^+ | \delta_C^+ \in i_C^+(x)\}, f_C^+(x) = \{\eta_C^+ | \eta_C^+ \in i_C^+(x)\}$
are three positive membership functions expressed by a few close intervals in the real unit interval [0,1] which represent the truth positive membership hesitant degree, indeterminacy positive membership hesitant degree and falsity positive membership hesitant degree, and:
$t_A^-(x) = \{\gamma_A^- | \gamma_A^- \in t_A^-(x)\}, i_A^-(x) = \{\delta_A^- | \delta_A^- \in i_A^-(x)\}, f_A^-(x) = \{\eta_A^- | \eta_A^- \in i_A^-(x)\}$
$t_B^-(x) = \{\gamma_B^- | \gamma_B^- \in t_B^-(x)\}, i_B^-(x) = \{\delta_B^- | \delta_B^- \in i_B^-(x)\}, f_B^-(x) = \{\eta_B^- | \eta_B^- \in i_B^-(x)\}$
$t_C^-(x) = \{\gamma_C^- | \gamma_C^- \in t_C^-(x)\}, i_C^-(x) = \{\delta_C^- | \delta_C^- \in i_C^-(x)\}, f_C^-(x) = \{\eta_C^- | \eta_C^- \in i_C^-(x)\}$
are three negative membership functions expressed by a few close intervals in the real unit interval $[-1,0]$ which represent the truth negative membership hesitant degree, indeterminacy negative membership hesitant degree and falsity negative membership hesitant degree.

4. PROPERTIES OF 2HZS

4.1 Complement

Definition 4

Let A be a 2HZS, the complement of a 2HZS A is denoted A^C and is defined by:

$$A^C = \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-}} \left\{ \left\langle \begin{array}{c} [\gamma_{LA}^+(x), \gamma_{UA}^+(x)], \\ [1 - \delta_{LA}^+(x), 1 - \delta_{UA}^+(x)], \\ [\eta_{LA}^+(x), \eta_{UA}^+(x)], \\ [\gamma_{LA}^-(x), \gamma_{UA}^-(x)], \\ [1 - \delta_{LA}^-(x), 1 - \delta_{UA}^-(x)], \\ [\eta_{LA}^-(x), \eta_{UA}^-(x)] \end{array} \right\rangle \right\} \quad (6)$$

4.2 Intersection

Definition 5

Let A, and B be two 2HZS, the intersection of two 2HZS A and B is denoted $A \cap B$and is defined by:

$$A \cap B = \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-, \\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+, \\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-}} \left\{ \begin{array}{c} \left[\bigwedge(\gamma_{LA}^+, \gamma_{LB}^+), \bigwedge(\gamma_{UA}^+, \gamma_{UB}^+)\right], \\ \left[\bigvee(\delta_{LA}^+, \delta_{LB}^+), \bigvee(\delta_{UA}^+, \delta_{UB}^+)\right], \\ \left[\bigvee(\eta_{LA}^+, \eta_{LB}^+), \bigvee(\eta_{UA}^+, \eta_{UB}^+)\right], \\ \left[\bigwedge(\gamma_{LA}^-, \gamma_{LB}^-), \bigwedge(\gamma_{UA}^-, \gamma_{UB}^-)\right], \\ \left[\bigvee(\delta_{LA}^-, \delta_{LB}^-), \bigvee(\delta_{UA}^-, \delta_{UB}^-)\right], \\ \left[\bigvee(\eta_{LA}^-, \eta_{LB}^-), \bigvee(\eta_{UA}^-, \eta_{UB}^-)\right] \end{array} \right\} \quad (7)$$

4.3 Union

Definition 6

Let A and B be two 2HZS, the union of a two 2HZS A and B is denoted $A \cup B$and is defined by:

$$A \cup B = \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-, \\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+, \\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-}} \left\{ \begin{array}{c} \left[\bigvee(\gamma_{LA}^+, \gamma_{LB}^+), \bigvee(\gamma_{UA}^+, \gamma_{UB}^+)\right], \\ \left[\bigwedge(\delta_{LA}^+, \delta_{LB}^+), \bigwedge(\delta_{UA}^+, \delta_{UB}^+)\right], \\ \left[\bigwedge(\eta_{LA}^+, \eta_{LB}^+), \bigwedge(\eta_{UA}^+, \eta_{UB}^+)\right], \\ \left[\bigvee(\gamma_{LA}^-, \gamma_{LB}^-), \bigvee(\gamma_{UA}^-, \gamma_{UB}^-)\right], \\ \left[\bigwedge(\delta_{LA}^-, \delta_{LB}^-), \bigwedge(\delta_{UA}^-, \delta_{UB}^-)\right], \\ \left[\bigwedge(\eta_{LA}^-, \eta_{LB}^-), \bigwedge(\eta_{UA}^-, \eta_{UB}^-)\right] \end{array} \right\} \quad (8)$$

5. OPERATIONS OF 2HZS

5.1 Sum algebraic

Definition 7

Let A and B be two 2HZS, the sum algebraic is defined as follows:

$$A + B = \cup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-, \\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+, \\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-}} \left\{ \begin{array}{c} [\gamma_{LA}^+ + \gamma_{LB}^+ - \gamma_{LA}^+\gamma_{LB}^+, \gamma_{UA}^+ + \gamma_{UB}^+ - \gamma_{UA}^+\gamma_{UB}^+], \\ [\delta_{LA}^+\delta_{LB}^+, \delta_{UA}^+\delta_{UB}^+], [\eta_{LA}^+\eta_{LB}^+, \eta_{UA}^+\eta_{UB}^+], \\ [\gamma_{LA}^- + \gamma_{LB}^- - \gamma_{LA}^-\gamma_{LB}^-, \gamma_{UA}^- + \gamma_{UB}^- - \gamma_{UA}^-\gamma_{UB}^-], \\ [\delta_{LA}^-\delta_{LB}^-, \delta_{UA}^-\delta_{UB}^-], [\eta_{LA}^-\eta_{LB}^-, \eta_{UA}^-\eta_{UB}^-] \end{array} \right\} \quad (9)$$

Theorem 1

Let A, B and C, be three 2HZS, the following equations are true:

(i) $A + B = B + A$ (10)

(ii) $(A + B) + C = A + (B + C)$ (11)

Proofs

(i) $B + A$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+,\\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-,\\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+,\\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-}} \left\{\begin{matrix} [\gamma_{LB}^+ + \gamma_{LA}^+ - \gamma_{LB}^+\gamma_{LA}^+, \gamma_{UB}^+ + \gamma_{UA}^+ - \gamma_{UB}^+\gamma_{UA}^+], \\ [\delta_{LB}^+\delta_{LA}^+, \delta_{UB}^+\delta_{UA}^+], [\eta_{LB}^+\eta_{LA}^+, \eta_{UB}^+\eta_{UA}^+], \\ [\gamma_{LB}^- + \gamma_{LA}^- - \gamma_{LB}^-\gamma_{LA}^-, \gamma_{UB}^- + \gamma_{UA}^- - \gamma_{UB}^-\gamma_{UA}^-], \\ [\delta_{LB}^-\delta_{LA}^-, \delta_{UB}^-\delta_{UA}^-], [\eta_{LB}^-\eta_{LA}^-, \eta_{UB}^-\eta_{UA}^-] \end{matrix}\right\}$$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+,\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-,\\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+,\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-}} \left\{\begin{matrix} \begin{bmatrix} \gamma_{LA}^+ + \gamma_{LB}^+ - \gamma_{LA}^+\gamma_{LB}^+, \\ \gamma_{UA}^+ + \gamma_{UB}^+ - \gamma_{UA}^+\gamma_{UB}^+ \end{bmatrix}, \\ [\delta_{LA}^+\delta_{LB}^+, \delta_{UA}^+\delta_{UB}^+], [\eta_{LA}^+\eta_{LB}^+, \eta_{UA}^+\eta_{UB}^+], \\ \begin{bmatrix} \gamma_{LA}^- + \gamma_{LB}^- - \gamma_{LA}^-\gamma_{LB}^-, \\ \gamma_{UA}^- + \gamma_{UB}^- - \gamma_{UA}^-\gamma_{UB}^- \end{bmatrix}, \\ [\delta_{LA}^-\delta_{LB}^-, \delta_{UA}^-\delta_{UB}^-], [\eta_{LA}^-\eta_{LB}^-, \eta_{UA}^-\eta_{UB}^-] \end{matrix}\right\}$$

$$= A + B$$

(ii) $(A + B) + C$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+,\\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-,\\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+,\\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-\\ \gamma_A^+ \in t_C^+,\ \delta_C^+ \in i_C^+, \eta_C^+ \in f_C^+,\\ \gamma_C^- \in t_C^-,\ \delta_C^- \in i_C^-, \eta_C^- \in f_C^-}} \left\langle\begin{matrix} [\gamma_{LA}^+ + \gamma_{LB}^+ - \gamma_{LA}^+\gamma_{LB}^+, \gamma_{UA}^+ + \gamma_{UB}^+ - \gamma_{UA}^+\gamma_{UB}^+], \\ [\delta_{LA}^+\delta_{LB}^+, \delta_{UA}^+\delta_{UB}^+], [\eta_{LA}^+\eta_{LB}^+, \eta_{UA}^+\eta_{UB}^+], \\ [\gamma_{LA}^- + \gamma_{LB}^- - \gamma_{LA}^-\gamma_{LB}^-, \gamma_{UA}^- + \gamma_{UB}^- - \gamma_{UA}^-\gamma_{UB}^-], \\ [\delta_{LA}^-\delta_{LB}^-, \delta_{UA}^-\delta_{UB}^-], [\eta_{LA}^-\eta_{LB}^-, \eta_{UA}^-\eta_{UB}^-] \end{matrix}\right\rangle + C$$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+,\\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-,\\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+,\\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-\\ \gamma_A^+ \in t_C^+,\ \delta_C^+ \in i_C^+, \eta_C^+ \in f_C^+,\\ \gamma_C^- \in t_C^-,\ \delta_C^- \in i_C^-, \eta_C^- \in f_C^-}} \left\langle\begin{matrix} [1 - (1 - \gamma_{LA}^+)(1 - \gamma_{LB}^+)], \\ [\delta_{LA}^+\delta_{LB}^+, \delta_{UA}^+\delta_{UB}^+], [\eta_{LA}^+\eta_{LB}^+, \eta_{UA}^+\eta_{UB}^+], \\ [1 - (1 - \gamma_{LA}^-)(1 - \gamma_{LB}^-)], \\ [\delta_{LA}^-\delta_{LB}^-, \delta_{UA}^-\delta_{UB}^-], [\eta_{LA}^-\eta_{LB}^-, \eta_{UA}^-\eta_{UB}^-] \end{matrix}\right\rangle + C$$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+,\\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-,\\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+,\\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-\\ \gamma_A^+ \in t_C^+,\ \delta_C^+ \in i_C^+, \eta_C^+ \in f_C^+,\\ \gamma_C^- \in t_C^-,\ \delta_C^- \in i_C^-, \eta_C^- \in f_C^-}} \begin{matrix} [1 - \{1 - [1 - (1 - \gamma_{LA}^+)(1 - \gamma_{LB}^+)]\}(1 - \gamma_{LC}^+)], \\ [\delta_{LA}^+\delta_{LB}^+\delta_{LC}^+, \delta_{UA}^+\delta_{UB}^+\delta_{UC}^+], [\eta_{LA}^+\eta_{LB}^+\eta_{LC}^+, \eta_{UA}^+\eta_{UB}^+\eta_{UC}^+], \\ [1 - \{1 - [1 - (1 - \gamma_{LA}^-)(1 - \gamma_{LB}^-)]\}(1 - \gamma_{LC}^-)], \\ [\delta_{LA}^-\delta_{LB}^-\delta_{LC}^-, \delta_{UA}^-\delta_{UB}^-\delta_{UC}^-], [\eta_{LA}^-\eta_{LB}^-\eta_{LC}^-, \eta_{UA}^-\eta_{UB}^-\eta_{UC}^-] \end{matrix}$$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+,\\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-,\\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+,\\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-\\ \gamma_A^+ \in t_C^+,\ \delta_C^+ \in i_C^+, \eta_C^+ \in f_C^+,\\ \gamma_C^- \in t_C^-,\ \delta_C^- \in i_C^-, \eta_C^- \in f_C^-}} \begin{matrix} \big[[1 - (1 - \gamma_{LA}^+)(1 - \gamma_{LB}^+)(1 - \gamma_{LC}^+)\big], [\delta_{LA}^+\delta_{LB}^+\delta_{LC}^+, \delta_{UA}^+\delta_{UB}^+\delta_{UC}^+], \\ [\eta_{LA}^+\eta_{LB}^+\eta_{LC}^+, \eta_{UA}^+\eta_{UB}^+\eta_{UC}^+] \\ \big[[1 - (1 - \gamma_{LA}^-)(1 - \gamma_{LB}^-)(1 - \gamma_{LC}^-)\big], [\delta_{LA}^-\delta_{LB}^-\delta_{LC}^-, \delta_{UA}^-\delta_{UB}^-\delta_{UC}^-], \\ [\eta_{LA}^-\eta_{LB}^-\eta_{LC}^-, \eta_{UA}^-\eta_{UB}^-\eta_{UC}^-] \end{matrix}$$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+,\\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-,\\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+,\\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-\\ \gamma_A^+ \in t_C^+,\ \delta_C^+ \in i_C^+, \eta_C^+ \in f_C^+,\\ \gamma_C^- \in t_C^-,\ \delta_C^- \in i_C^-, \eta_C^- \in f_C^-}} \begin{matrix} [1 - (1 - \gamma_{LA}^+)\{1 - [1 - (1 - \gamma_{LB}^+)(1 - \gamma_{LC}^+)]\}], \\ [\delta_{LA}^+\delta_{LB}^+\delta_{LC}^+, \delta_{UA}^+\delta_{UB}^+\delta_{UC}^+], \\ [\eta_{LA}^+\eta_{LB}^+\eta_{LC}^+, \eta_{UA}^+\eta_{UB}^+\eta_{UC}^+], \\ [1 - (1 - \gamma_{LA}^-)\{1 - [1 - (1 - \gamma_{LB}^-)(1 - \gamma_{LC}^-)]\}], \\ [\delta_{LA}^-\delta_{LB}^-\delta_{LC}^-, \delta_{UA}^-\delta_{UB}^-\delta_{UC}^-], [\eta_{LA}^-\eta_{LB}^-\eta_{LC}^-, \eta_{UA}^-\eta_{UB}^-\eta_{UC}^-] \end{matrix}$$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-, \\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+, \\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^- \\ \gamma_A^+ \in t_C^+,\ \delta_C^+ \in i_C^+, \eta_C^+ \in f_C^+, \\ \gamma_C^- \in t_C^-,\ \delta_C^- \in i_C^-, \eta_C^- \in f_C^-}} A + \begin{pmatrix} [1-(1-\gamma_{LB}^+)(1-\gamma_{LC}^+)], \\ [\delta_{LB}^+\delta_{LC}^+, \delta_{UB}^+\delta_{UC}^+], [\eta_{LB}^+\eta_{LC}^+, \eta_{UB}^+\eta_{UC}^+], \\ [1-(1-\gamma_{LB}^-)(1-\gamma_{LC}^-)], \\ [\delta_{LB}^-\delta_{LC}^-, \delta_{UB}^-\delta_{UC}^-], \\ [\eta_{LB}^-\eta_{LC}^-, \eta_{UB}^-\eta_{UC}^-] \end{pmatrix}$$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-, \\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+, \\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-, \\ \gamma_A^+ \in t_C^+,\ \delta_C^+ \in i_C^+, \eta_C^+ \in f_C^+, \\ \gamma_C^- \in t_C^-,\ \delta_C^- \in i_C^-, \eta_C^- \in f_C^-}} A + \begin{pmatrix} [\gamma_{LB}^+ + \gamma_{LC}^+ - \gamma_{LB}^+\gamma_{LC}^+, \gamma_{UB}^+ + \gamma_{UC}^+ - \gamma_{UB}^+\gamma_{UC}^+], \\ [\delta_{LB}^+\delta_{LC}^+, \delta_{UB}^+\delta_{UC}^+], [\eta_{LB}^+\eta_{LC}^+, \eta_{UB}^+\eta_{UC}^+], \\ [\gamma_{LB}^- + \gamma_{LC}^- - \gamma_{LB}^-\gamma_{LC}^-, \gamma_{UB}^- + \gamma_{UC}^- - \gamma_{UB}^-\gamma_{UC}^-], \\ [\delta_{LB}^-\delta_{LC}^-, \delta_{UB}^-\delta_{UC}^-], [\eta_{LB}^-\eta_{LC}^-, \eta_{UB}^-\eta_{UC}^-] \end{pmatrix}$$

$$= A + (B + C)$$

□

5.2 Difference

Definition 8

Let A and B be two 2HZS, the difference is defined as follows :

$$A \ominus B = \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-, \\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+, \\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-}} \left\{ \begin{matrix} [\gamma_{L(A\ominus B)}^+, \gamma_{U(A\ominus B)}^+], [\delta_{L(A\ominus B)}^+, \delta_{U(A\ominus B)}^+], \\ [\eta_{L(A\ominus B)}^+, \eta_{U(A\ominus B)}^+], [\gamma_{L(A\ominus B)}^-, \gamma_{U(A\ominus B)}^-], \\ [\delta_{L(A\ominus B)}^-, \delta_{U(A\ominus B)}^-], [\eta_{L(A\ominus B)}^-, \eta_{U(A\ominus B)}^-] \end{matrix} \right\} \quad (12)$$

Where
$\gamma_{L(A\ominus B)}^+ = \wedge\ \gamma_{LA}^+(x), \eta_{LB}^+(x)$, $\delta_{L(A\ominus B)}^+ = \vee\ \delta_{LA}^+(x), 1-\delta_{UB}^+(x)$, $\eta_{L(A\ominus B)}^+ = \vee\ \eta_{LA}^+(x), \gamma_{LB}^+(x)$,
$\gamma_{U(A\ominus B)}^+ = \wedge\ \gamma_{UA}^+(x), \eta_{UB}^+(x)$, $\delta_{U(A\ominus B)}^+ = \vee\ \delta_{UB}^+(x), 1-\delta_{LB}^+(x)$, $\eta_{U(A\ominus B)}^+ = \vee\ \eta_{UA}^+(x), \gamma_{UB}^+(x)$,
$\gamma_{L(A\ominus B)}^- = \wedge\ \gamma_{LA}^-(x), \eta_{LB}^-(x))$,$\delta_{L(A\ominus B)}^- = \vee\ \delta_{LA}^-(x), 1-\delta_{UB}^-(x)(x)$, $\eta_{L(A\ominus B)}^- = \vee\ \eta_{LA}^-(x), \gamma_{LB}^-(x)$,
$\gamma_{U(A\ominus B)}^- = \wedge\ \gamma_{UA}^-(x), \eta_{UB}^-(x)$,$\delta_{U(A\ominus B)}^- = \vee\ \delta_{UB}^-(x), 1-\delta_{LB}^-(x)$, $\eta_{U(A\ominus B)}^- = \vee\ \eta_{UA}^-(x), \gamma_{UB}^-(x)$.

5.3 Product

Definition 9

Let A and B be two 2HZS, the product is defined as follows :

$$A\,.\,B = \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-, \\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+, \\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-}} \left\{ \begin{matrix} [\gamma_{LA}^+\gamma_{LB}^+, \gamma_{UA}^+\gamma_{UB}^+], \begin{bmatrix} \delta_{LA}^+ + \delta_{LB}^+ - \delta_{LA}^+\delta_{LB}^+, \\ \delta_{UA}^+ + \delta_{UB}^+ - \delta_{UA}^+\delta_{UB}^+ \end{bmatrix}, \\ \begin{bmatrix} \eta_{LA}^+ + \eta_{LB}^+ - \eta_{LA}^+\eta_{LB}^+, \\ \eta_{UA}^+ + \eta_{UB}^+ - \eta_{UA}^+\eta_{UB}^+ \end{bmatrix}, \\ [\gamma_{LA}^-\gamma_{LB}^-, \gamma_{UA}^-\gamma_{UB}^-], \begin{bmatrix} \delta_{LA}^- + \delta_{LB}^- - \delta_{LA}^-\delta_{LB}^-, \\ \delta_{UA}^- + \delta_{UB}^- - \delta_{UA}^-\delta_{UB}^- \end{bmatrix}, \begin{bmatrix} \eta_{LA}^- + \eta_{LB}^- - \eta_{LA}^-\eta_{LB}^-, \\ \eta_{UA}^- + \eta_{UB}^- - \eta_{UA}^-\eta_{UB}^- \end{bmatrix} \end{matrix} \right\} \quad (13)$$

Theorem 2

Let A, B and C be three 2HZS, the following equations are true:

(1) $A \cdot B = B \cdot A$ (14)
(2) $(A \cdot B) \cdot C = A \cdot (B \cdot C)$ (15)

Proofs

(i) $B \cdot A$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-, \\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+, \\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-}} \left\{ \begin{matrix} [\gamma_{LB}^+\gamma_{LA}^+, \gamma_{UB}^+\gamma_{UA}^+], [\delta_{LB}^+ + \delta_{LA}^+ - \delta_{LB}^+\delta_{LA}^+, \delta_{UB}^+ + \delta_{UA}^+ - \delta_{UB}^+\delta_{UA}^+], \\ [\eta_{LB}^+ + \eta_{LA}^+ - \eta_{LB}^+\eta_{LA}^+, \eta_{UB}^+ + \eta_{UA}^+ - \eta_{UB}^+\eta_{UA}^+], \\ [\gamma_{LB}^-\gamma_{LA}^-, \gamma_{UB}^-\gamma_{UA}^-], [\delta_{LB}^- + \delta_{LA}^- - \delta_{LB}^-\delta_{LA}^-, \delta_{UB}^- + \delta_{UA}^- - \delta_{UB}^-\delta_{UA}^-], \\ [\eta_{LB}^- + \eta_{LA}^- - \eta_{LB}^-\eta_{LA}^-, \eta_{UB}^- + \eta_{UA}^- - \eta_{UB}^-\eta_{UA}^-] \end{matrix} \right\}$$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-, \\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+, \\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-}} \left\{ \begin{matrix} [\gamma_{LA}^+\gamma_{LB}^+, \gamma_{UA}^+\gamma_{UB}^+], \\ [\delta_{LA}^+ + \delta_{LB}^+ - \delta_{LA}^+\delta_{LB}^+, \delta_{UA}^+ + \delta_{UB}^+ - \delta_{UA}^+\delta_{UB}^+], \\ [\eta_{LA}^+ + \eta_{LB}^+ - \eta_{LA}^+\eta_{LB}^+, \eta_{UA}^+ + \eta_{UB}^+ - \eta_{UA}^+\eta_{UB}^+], \\ [\gamma_{LA}^-\gamma_{LB}^-, \gamma_{UA}^-\gamma_{UB}^-], \\ [\delta_{LA}^- + \delta_{LB}^- - \delta_{LA}^-\delta_{LB}^-, \delta_{UA}^- + \delta_{UB}^- - \delta_{UA}^-\delta_{UB}^-], \\ [\eta_{LA}^- + \eta_{LB}^- - \eta_{LA}^-\eta_{LB}^-, \eta_{UA}^- + \eta_{UB}^- - \eta_{UA}^-\eta_{UB}^-] \end{matrix} \right\}$$

$$= A \cdot B$$

(ii)$(A \,.\, B\,).\,C$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-, \\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+, \\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-, \\ \gamma_A^+ \in t_C^+,\ \delta_C^+ \in i_C^+, \eta_C^+ \in f_C^+, \\ \gamma_C^- \in t_C^-,\ \delta_C^- \in i_C^-, \eta_C^- \in f_C^-}} \left(\begin{matrix} [\gamma_{LA}^+\gamma_{LB}^+, \gamma_{UA}^+\gamma_{UB}^+], \\ [\delta_{LA}^+ + \delta_{LB}^+ - \delta_{LA}^+\delta_{LB}^+, \delta_{UA}^+ + \delta_{UB}^+ - \delta_{UA}^+\delta_{UB}^+], \\ [\eta_{LA}^+ + \eta_{LB}^+ - \eta_{LA}^+\eta_{LB}^+, \eta_{UA}^+ + \eta_{UB}^+ - \eta_{UA}^+\eta_{UB}^+], \\ [\gamma_{LA}^-\gamma_{LB}^-, \gamma_{UA}^-\gamma_{UB}^-], \\ [\delta_{LA}^- + \delta_{LB}^- - \delta_{LA}^-\delta_{LB}^-, \delta_{UA}^- + \delta_{UB}^- - \delta_{UA}^-\delta_{UB}^-], \\ [\eta_{LA}^- + \eta_{LB}^- - \eta_{LA}^-\eta_{LB}^-, \eta_{UA}^- + \eta_{UB}^- - \eta_{UA}^-\eta_{UB}^-] \end{matrix} \right) . C$$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-, \\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+, \\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-, \\ \gamma_A^+ \in t_C^+,\ \delta_C^+ \in i_C^+, \eta_C^+ \in f_C^+, \\ \gamma_C^- \in t_C^-,\ \delta_C^- \in i_C^-, \eta_C^- \in f_C^-}} \left(\begin{matrix} [\gamma_{LA}^+\gamma_{LB}^+, \gamma_{UA}^+\gamma_{UB}^+], [1-(1-\delta_{LA}^+)(1-\delta_{LB}^+)], \\ [1-(1-\eta_{LA}^+)(1-\eta_{LB}^+)] \\ [\gamma_{LA}^-\gamma_{LB}^-, \gamma_{UA}^-\gamma_{UB}^-], [1-(1-\delta_{LA}^-)(1-\delta_{LB}^-)], \\ [1-(1-\eta_{LA}^-)(1-\eta_{LB}^-)] \end{matrix} \right) . C$$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-, \\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+, \\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-, \\ \gamma_A^+ \in t_C^+,\ \delta_C^+ \in i_C^+, \eta_C^+ \in f_C^+, \\ \gamma_C^- \in t_C^-,\ \delta_C^- \in i_C^-, \eta_C^- \in f_C^-}} \left(\begin{matrix} [\gamma_{LA}^+\gamma_{LB}^+\gamma_{LC}^+, \gamma_{UA}^+\gamma_{UB}^+\gamma_{UC}^+], \\ [1-\{1-[1-(1-\delta_{LA}^+)(1-\delta_{LB}^+)]\}(1-\delta_{LC}^+)], \\ [1-\{1-[1-(1-\eta_{LA}^+)(1-\eta_{LB}^+)]\}(1-\eta_{LC}^+)], \\ [\gamma_{LA}^-\gamma_{LB}^-\gamma_{LC}^-, \gamma_{UA}^-\gamma_{UB}^-\gamma_{UC}^-], \\ [1-\{1-[1-(1-\delta_{LA}^-)(1-\delta_{LB}^-)]\}(1-\delta_{LC}^-)], \\ [1-\{1-[1-(1-\eta_{LA}^-)(1-\eta_{LB}^-)]\}(1-\eta_{LC}^-)] \end{matrix} \right)$$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-, \\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+, \\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-, \\ \gamma_A^+ \in t_C^+,\ \delta_C^+ \in i_C^+, \eta_C^+ \in f_C^+, \\ \gamma_C^- \in t_C^-,\ \delta_C^- \in i_C^-, \eta_C^- \in f_C^-}} \begin{pmatrix} [\gamma_{LA}^+\gamma_{LB}^+\gamma_{LC}^+, \gamma_{UA}^+\gamma_{UB}^+\gamma_{UC}^+], \\ [1-(1-\delta_{LA}^+)(1-\delta_{LB}^+)(1-\delta_{LC}^+)], \\ [1-(1-\eta_{LA}^+)(1-\eta_{LB}^+)(1-\eta_{LC}^+)], \\ [\gamma_{LA}^-\gamma_{LB}^-\gamma_{LC}^-, \gamma_{UA}^-\gamma_{UB}^-\gamma_{UC}^-], \\ [1-(1-\delta_{LA}^-)(1-\delta_{LB}^-)(1-\delta_{LC}^-)], \\ [1-(1-\eta_{LA}^-)(1-\eta_{LB}^-)(1-\eta_{LC}^-)] \end{pmatrix}$$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-, \\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+, \\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-, \\ \gamma_A^+ \in t_C^+,\ \delta_C^+ \in i_C^+, \eta_C^+ \in f_C^+, \\ \gamma_C^- \in t_C^-,\ \delta_C^- \in i_C^-, \eta_C^- \in f_C^-}} \begin{pmatrix} [\gamma_{LA}^+\gamma_{LB}^+\gamma_{LC}^+, \gamma_{UA}^+\gamma_{UB}^+\gamma_{UC}^+], \\ [1-(1-\delta_{LA}^+)\{1-[1-(1-\delta_{LB}^+)(1-\delta_{LC}^+)]\}], \\ [1-(1-\eta_{LA}^+)\{1-[1-(1-\eta_{LB}^+)(1-\eta_{LC}^+)]\}], \\ [\gamma_{LA}^-\gamma_{LB}^-\gamma_{LC}^-, \gamma_{UA}^-\gamma_{UB}^-\gamma_{UC}^-], \\ [1-(1-\delta_{LA}^-)\{1-[1-(1-\delta_{LB}^-)(1-\delta_{LC}^-)]\}], \\ [1-(1-\eta_{LA}^-)\{1-[1-(1-\eta_{LB}^-)(1-\eta_{LC}^-)]\}] \end{pmatrix}$$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-, \\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+, \\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-, \\ \gamma_A^+ \in t_C^+,\ \delta_C^+ \in i_C^+, \eta_C^+ \in f_C^+, \\ \gamma_C^- \in t_C^-,\ \delta_C^- \in i_C^-, \eta_C^- \in f_C^-}} A\,.\begin{pmatrix} [\gamma_{LB}^+\gamma_{LC}^+, \gamma_{UB}^+\gamma_{UC}^+], \\ [1-(1-\delta_{LB}^+)(1-\delta_{LC}^+)], \\ [1-(1-\eta_{LB}^+)(1-\eta_{LC}^+)], \\ [\gamma_{LB}^-\gamma_{LC}^-, \gamma_{UB}^-\gamma_{UC}^-], \\ [1-(1-\delta_{LB}^-)(1-\delta_{LC}^-)], \\ [1-(1-\eta_{LB}^-)(1-\eta_{LC}^-)] \end{pmatrix}$$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-, \\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+, \\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-, \\ \gamma_A^+ \in t_C^+,\ \delta_C^+ \in i_C^+, \eta_C^+ \in f_C^+, \\ \gamma_C^- \in t_C^-,\ \delta_C^- \in i_C^-, \eta_C^- \in f_C^-}} A\,.\begin{pmatrix} [\gamma_{LB}^+\gamma_{LC}^+, \gamma_{UB}^+\gamma_{UC}^+], \\ [\delta_{LB}^+ + \delta_{LC}^+ - \delta_{LB}^+\delta_{LC}^+, \delta_{UB}^+ + \delta_{UC}^+ - \delta_{UB}^+\delta_{UC}^+], \\ [\eta_{LB}^+ + \eta_{LC}^+ - \eta_{LB}^+\eta_{LC}^+, \eta_{UB}^+ + \eta_{UC}^+ - \eta_{UB}^+\eta_{UC}^+], \\ [\gamma_{LB}^-\gamma_{LC}^-, \gamma_{UB}^-\gamma_{UC}^-], \\ [\delta_{LB}^- + \delta_{LC}^- - \delta_{LB}^-\delta_{LC}^-, \delta_{UB}^- + \delta_{UC}^- - \delta_{UB}^-\delta_{UC}^-], \\ [\eta_{LB}^- + \eta_{LC}^- - \eta_{LB}^-\eta_{LC}^-, \eta_{UB}^- + \eta_{UC}^- - \eta_{UB}^-\eta_{UC}^-] \end{pmatrix}$$

$$= A\,.(B\,.C)$$

□

5.4 Scalar Product

Definition 10

Let $A = \langle [t_{LA}^+, t_{UA}^+], [i_{LA}^+, i_{UA}^+], [f_{LA}^+, f_{UA}^+], [t_{LA}^-, t_{UA}^-], [i_{LA}^-, i_{UA}^-], [f_{LA}^-, f_{UA}^-] \rangle$, be 2HZS, when $\lambda > 0$, the scalar multiplication of 2HZS A is λA, its truth positive membership hesitant degree, indeterminacy positive membership hesitant degree, falsity positive membership hesitant degree, truth negative membership hesitant degree, indeterminacy negative membership hesitant degree and falsity negative membership hesitant degree functions are related to those of those by:

$$\lambda A = \cup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-}} \begin{Bmatrix} [1-(1-\gamma_{LA}^+)^\lambda, 1-(1-\gamma_{UA}^+)^\lambda], \\ [(\delta_{LA}^+)^\lambda, (\delta_{UA}^+)^\lambda], [(\eta_{LA}^+)^\lambda, (\eta_{UA}^+)^\lambda], \\ [1-(1-\gamma_{LA}^-)^\lambda, 1-(1-\gamma_{UA}^-)^\lambda], \\ [(\delta_{LA}^-)^\lambda, (\delta_{UA}^-)^\lambda], [(\eta_{LA}^-)^\lambda, (\eta_{UA}^-)^\lambda] \end{Bmatrix} \tag{16}$$

Theorem 3

Let A, B, C be three 2HZS, $\lambda, \lambda_1, \lambda_2 > 0$, the following equations are true:

(i) $\lambda(A+B) \;=\; \lambda A+\; \lambda B$ (17)

(ii) $\lambda_1 A+\lambda_2 A=\;(\lambda_1+\lambda_2)A$ (18)

Proofs

(i) $\lambda(A+B)$

$$=\bigcup_{\substack{\gamma_A^+\in t_A^+,\ \delta_A^+\in i_A^+,\eta_A^+\in f_A^+,\\ \gamma_A^-\in t_A^-,\ \delta_A^-\in i_A^-,\eta_A^-\in f_A^-,\\ \gamma_B^+\in t_B^+,\ \delta_B^+\in i_B^+,\eta_B^+\in f_B^+,\\ \gamma_B^-\in t_B^-,\ \delta_B^-\in i_B^-,\eta_B^-\in f_B^-}} \lambda\left(\begin{array}{c}[\gamma_{LA}^+ +\gamma_{LB}^+ -\gamma_{LA}^+\gamma_{LB}^+,\gamma_{UA}^+ +\gamma_{UB}^+ -\gamma_{UA}^+\gamma_{UB}^+],\\ [\delta_{LA}^+\delta_{LB}^+,\delta_{UA}^+\delta_{UB}^+],\\ [\eta_{LA}^+\eta_{LB}^+,\eta_{UA}^+\eta_{UB}^+],\\ [\gamma_{LA}^- +\gamma_{LB}^- -\gamma_{LA}^-\gamma_{LB}^-,\gamma_{UA}^- +\gamma_{UB}^- -\gamma_{UA}^-\gamma_{UB}^-],\\ [\delta_{LA}^-\delta_{LB}^-,\delta_{UA}^-\delta_{UB}^-],\\ [\eta_{LA}^-\eta_{LB}^-,\eta_{UA}^-\eta_{UB}^-]\end{array}\right)$$

$$=\bigcup_{\substack{\gamma_A^+\in t_A^+,\ \delta_A^+\in i_A^+,\eta_A^+\in f_A^+,\\ \gamma_A^-\in t_A^-,\ \delta_A^-\in i_A^-,\eta_A^-\in f_A^-,\\ \gamma_B^+\in t_B^+,\ \delta_B^+\in i_B^+,\eta_B^+\in f_B^+,\\ \gamma_B^-\in t_B^-,\ \delta_B^-\in i_B^-,\eta_B^-\in f_B^-}} \left(\begin{array}{c}\begin{bmatrix}1-(1-\gamma_{LA}^+ +\gamma_{LB}^+ -\gamma_{LA}^+\gamma_{LB}^+)^\lambda,\\ 1-(1-\gamma_{UA}^+ +\gamma_{UB}^+ -\gamma_{UA}^+\gamma_{UB}^+)^\lambda\end{bmatrix},\\ \left[(\delta_{LA}^+\delta_{LB}^+)^\lambda,(\delta_{UA}^+\delta_{UB}^+)^\lambda\right],\\ \left[(\eta_{LA}^+\eta_{LB}^+)^\lambda,(\eta_{UA}^+\eta_{UB}^+)^\lambda\right],\\ \begin{bmatrix}1-(1-\gamma_{LA}^- +\gamma_{LB}^- -\gamma_{LA}^-\gamma_{LB}^-)^\lambda,\\ 1-(1-\gamma_{UA}^- +\gamma_{UB}^- -\gamma_{UA}^-\gamma_{UB}^-)^\lambda\end{bmatrix},\\ \left[(\delta_{LA}^-\delta_{LB}^-)^\lambda,(\delta_{UA}^-\delta_{UB}^-)^\lambda\right],\\ \left[(\eta_{LA}^-\eta_{LB}^-)^\lambda,(\eta_{UA}^-\eta_{UB}^-)^\lambda\right]\end{array}\right)$$

$$=\bigcup_{\substack{\gamma_A^+\in t_A^+,\ \delta_A^+\in i_A^+,\eta_A^+\in f_A^+,\ \gamma_A^-\in t_A^-,\ \delta_A^-\in i_A^-,\eta_A^-\in f_A^-,\\ \gamma_B^+\in t_B^+,\ \delta_B^+\in i_B^+,\eta_B^+\in f_B^+,\ \gamma_B^-\in t_B^-,\ \delta_B^-\in i_B^-,\eta_B^-\in f_B^-}} \left\langle\begin{array}{c}\begin{bmatrix}1-(1-\gamma_{LA}^+)^\lambda(1-\gamma_{LB}^+)^\lambda,\\ 1-(1-\gamma_{UA}^+)^\lambda(1-\gamma_{UB}^+)^\lambda\end{bmatrix},\\ \left[(\delta_{LA}^+\delta_{LB}^+)^\lambda,(\delta_{UA}^+\delta_{UB}^+)^\lambda\right],\\ \left[(\eta_{LA}^+\eta_{LB}^+)^\lambda,(\eta_{UA}^+\eta_{UB}^+)^\lambda\right],\\ \begin{bmatrix}1-(1-\gamma_{LA}^-)^\lambda(1-\gamma_{LB}^-)^\lambda,\\ 1-(1-\gamma_{UA}^-)^\lambda(1-\gamma_{UB}^-)^\lambda\end{bmatrix},\\ \left[(\delta_{LA}^-\delta_{LB}^-)^\lambda,(\delta_{UA}^-\delta_{UB}^-)^\lambda\right],\\ \left[(\eta_{LA}^-\eta_{LB}^-)^\lambda,(\eta_{UA}^-\eta_{UB}^-)^\lambda\right]\end{array}\right\rangle$$

$$=\bigcup_{\substack{\gamma_A^+\in t_A^+,\ \delta_A^+\in i_A^+,\eta_A^+\in f_A^+,\ \gamma_A^-\in t_A^-,\ \delta_A^-\in i_A^-,\eta_A^-\in f_A^-,\\ \gamma_B^+\in t_B^+,\ \delta_B^+\in i_B^+,\eta_B^+\in f_B^+,\ \gamma_B^-\in t_B^-,\ \delta_B^-\in i_B^-,\eta_B^-\in f_B^-}} \left\langle\begin{array}{c}\begin{bmatrix}1-\left(1-(1-\gamma_{LA}^+)^\lambda\right)\left(1-(1-\gamma_{LB}^+)^\lambda\right),\\ 1-\left(1-(1-\gamma_{UA}^+)^\lambda\right)\left(1-(1-\gamma_{UB}^+)^\lambda\right)\end{bmatrix},\\ \left[(\delta_{LA}^+)^\lambda(\delta_{LB}^+)^\lambda,(\delta_{UA}^+)^\lambda\,(\delta_{UB}^+)^\lambda\right],\\ \left[(\eta_{LA}^+)^\lambda(\eta_{LB}^+)^\lambda,(\eta_{UA}^+)^\lambda(\eta_{UB}^+)^\lambda\right],\\ \begin{bmatrix}1-\left(1-(1-\gamma_{LA}^-)^\lambda\right)\left(1-(1-\gamma_{LB}^-)^\lambda\right),\\ 1-\left(1-(1-\gamma_{UA}^-)^\lambda\right)\left(1-(1-\gamma_{UB}^-)^\lambda\right)\end{bmatrix},\\ \left[(\delta_{LA}^-)^\lambda(\delta_{LB}^-)^\lambda,(\delta_{UA}^-)^\lambda\,(\delta_{UB}^-)^\lambda\right],\\ \left[(\eta_{LA}^-)^\lambda(\eta_{LB}^-)^\lambda,(\eta_{UA}^-)^\lambda(\eta_{UB}^-)^\lambda\right]\end{array}\right\rangle$$

$$=\bigcup_{\substack{\gamma_A^+\in t_A^+,\ \delta_A^+\in i_A^+,\eta_A^+\in f_A^+,\ \gamma_A^-\in t_A^-,\ \delta_A^-\in i_A^-,\eta_A^-\in f_A^-,\\ \gamma_B^+\in t_B^+,\ \delta_B^+\in i_B^+,\eta_B^+\in f_B^+,\ \gamma_B^-\in t_B^-,\ \delta_B^-\in i_B^-,\eta_B^-\in f_B^-}} \left(\left\langle\begin{array}{c}\left[1-(1-\gamma_{LA}^+)^\lambda,1-(1-\gamma_{UA}^+)^\lambda\right],\\ \left[(\delta_{LA}^+)^\lambda,(\delta_{UA}^+)^\lambda\right],\left[(\eta_{LA}^+)^\lambda,(\eta_{UA}^+)^\lambda\right]\end{array}\right\rangle\right)$$
$$+\left(\left\langle\begin{array}{c}\left[1-(1-\gamma_{LB}^+)^\lambda,1-(1-\gamma_{UB}^+)^\lambda\right],\\ \left[(\delta_{LB}^+)^\lambda,(\delta_{UB}^+)^\lambda\right],\left[(\eta_{LB}^+)^\lambda,(\eta_{UB}^+)^\lambda\right]\end{array}\right\rangle\right)$$

$$=\;\lambda A+\;\lambda B$$

(ii)$\lambda_1 A+\lambda_2 A$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-}} \begin{pmatrix} [1-(1-\gamma_{LA}^+)^{\lambda_1}, 1-(1-\gamma_{UA}^+)^{\lambda_1}], \\ [(\delta_{LA}^+)^{\lambda_1}, (\delta_{UA}^+)^{\lambda_1}], [(\eta_{LA}^+)^{\lambda_1}, (\eta_{UA}^+)^{\lambda_1}], \\ [1-(1-\gamma_{LA}^-)^{\lambda_1}, 1-(1-\gamma_{UA}^-)^{\lambda_1}], \\ [(\delta_{LA}^-)^{\lambda_1}, (\delta_{UA}^-)^{\lambda_1}], [(\eta_{LA}^-)^{\lambda_1}, (\eta_{UA}^-)^{\lambda_1}] \end{pmatrix}$$

$$+ \begin{pmatrix} [1-(1-\gamma_{LA}^+)^{\lambda_2}, 1-(1-\gamma_{UA}^+)^{\lambda_2}], \\ [(\delta_{LA}^+)^{\lambda_2}, (\delta_{UA}^+)^{\lambda_2}], [(\eta_{LA}^+)^{\lambda_2}, (\eta_{UA}^+)^{\lambda_2}], \\ [1-(1-\gamma_{LA}^-)^{\lambda_2}, 1-(1-\gamma_{UA}^-)^{\lambda_2}], \\ [(\delta_{LA}^-)^{\lambda_2}, (\delta_{UA}^-)^{\lambda_2}], [(\eta_{LA}^-)^{\lambda_2}, (\eta_{UA}^-)^{\lambda_2}] \end{pmatrix}$$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-}} \left\langle \begin{matrix} \begin{bmatrix} 1-[1-(1-\gamma_{LA}^+)^{\lambda_1}],[1-(1-\gamma_{UA}^+)^{\lambda_2}], \\ 1-[1-(1-\gamma_{UA}^+)^{\lambda_1}],[1-(1-\gamma_{UA}^+)^{\lambda_2}] \end{bmatrix}, \\ [(\delta_{LA}^+)^{\lambda_1}(\delta_{LA}^+)^{\lambda_2}, (\delta)^{\lambda_1}(\delta_{UA}^+)^{\lambda_2}], \\ [(\eta_{LA}^+)^{\lambda_1}(\eta_{LA}^+)^{\lambda_2}, (\eta_{UA}^+)^{\lambda_1}(\eta_{UA}^+)^{\lambda_2}], \\ \begin{bmatrix} 1-[1-(1-\gamma_{LA}^-)^{\lambda_1}],[1-(1-\gamma_{UA}^-)^{\lambda_2}], \\ 1-[1-(1-\gamma_{UA}^-)^{\lambda_1}],[1-(1-\gamma_{UA}^-)^{\lambda_2}] \end{bmatrix}, \\ [(\delta_{LA}^-)^{\lambda_1}(\delta_{LA}^-)^{\lambda_2}, (\delta_{UA}^-)^{\lambda_1}(\delta_{UA}^-)^{\lambda_2}], [(\eta_{LA}^-)^{\lambda_1}(\eta_{LA}^-)^{\lambda_2}, (\eta_{UA}^-)^{\lambda_1}(\eta_{UA}^-)^{\lambda_2}] \end{matrix} \right\rangle$$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-}} \begin{pmatrix} \begin{bmatrix} 1-(1-\gamma_{LA}^+)^{\lambda_1}(1-\gamma_{LA}^+)^{\lambda_2}, \\ 1-(1-\gamma_{UA}^+)^{\lambda_1}(1-\gamma_{UA}^+)^{\lambda_2} \end{bmatrix}, \\ [(\delta_{LA}^+)^{\lambda_1}(\delta_{LA}^+)^{\lambda_2}, (\delta_{UA}^+)^{\lambda_1}(\delta_{UA}^+)^{\lambda_2}], \\ [(\eta_{LA}^+)^{\lambda_1}(\eta_{LA}^+)^{\lambda_2}, (\eta_{UA}^+)^{\lambda_1}(\eta_{UA}^+)^{\lambda_2}], \\ \begin{bmatrix} 1-(1-\gamma_{LA}^-)^{\lambda_1}(1-\gamma_{LA}^-)^{\lambda_2}, \\ 1-(1-\gamma_{UA}^-)^{\lambda_1}(1-\gamma_{UA}^-)^{\lambda_2} \end{bmatrix}, \\ [(\delta_{LA}^-)^{\lambda_1}(\delta_{LA}^-)^{\lambda_2}, (\delta_{UA}^-)^{\lambda_1}(\delta_{UA}^-)^{\lambda_2}], \\ [(\eta_{LA}^-)^{\lambda_1}(\eta_{LA}^-)^{\lambda_2}, (\eta_{UA}^-)^{\lambda_1}(\eta_{UA}^-)^{\lambda_2}] \end{pmatrix}$$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-}} \begin{pmatrix} [1-(1-\gamma_{LA}^+)^{(\lambda_1+\lambda_2)}, 1-(1-\gamma_{UA}^+)^{(\lambda_1+\lambda_2)}], \\ [(\delta_{LA}^+)^{(\lambda_1+\lambda_2)}, (\delta_{UA}^+)^{(\lambda_1+\lambda_2)}], \\ [(\eta_{LA}^+)^{(\lambda_1+\lambda_2)}, (\eta_{UA}^+)^{(\lambda_1+\lambda_2)}], \\ [1-(1-\gamma_{LA}^-)^{(\lambda_1+\lambda_2)}, 1-(1-\gamma_{UA}^-)^{(\lambda_1+\lambda_2)}], \\ [(\delta_{LA}^-)^{(\lambda_1+\lambda_2)}, (\delta_{UA}^-)^{(\lambda_1+\lambda_2)}], \\ [(\eta_{LA}^-)^{(\lambda_1+\lambda_2)}, (\eta_{UA}^-)^{(\lambda_1+\lambda_2)}] \end{pmatrix}$$

$$= (\lambda_1 + \lambda_2)A$$

□

5.5 Scalar division

Definition 11

Let A be 2HZS, when $\lambda > 0$, the scalar division of the interval valued bipolar neutrosophic set A is A/λ:

$$A/\lambda = \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-}} \left\{ \begin{matrix} [1-(1-\gamma_{LA}^+)/\lambda, 1-(1-\gamma_{UA}^+)/\lambda], \\ [(\delta_{LA}^+)/\lambda, (\delta_{UA}^+)/\lambda], [(\eta_{LA}^+)/\lambda, (\eta_{UA}^+)/\lambda], \\ [1-(1-\gamma_{LA}^-)/\lambda, 1-(1-\gamma_{UA}^-)/\lambda], \\ [(\delta_{LA}^-)/\lambda, (\delta_{UA}^-)/\lambda], [(\eta_{LA}^-)/\lambda, (\eta_{UA}^-)/\lambda] \end{matrix} \right\} \tag{19}$$

5.6 Scalar Power

Definition 12

Let A be 2HZS, when $\lambda > 0$, the scalar power of 2HZS A is A^λ:

$$A^\lambda = \cup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-}} \left\{ \begin{matrix} [(\gamma_{LA}^+)^\lambda, (\gamma_{UA}^+)^\lambda], [1-(1-\delta_{LA}^+)^\lambda, 1-(1-\delta_{UA}^+)^\lambda], \\ [1-(1-\eta_{LA}^+)^\lambda, 1-(1-\eta_{UA}^+)^\lambda], [(\gamma_{LA}^-)^\lambda, (\gamma_{UA}^-)^\lambda], \\ [1-(1-\delta_{LA}^-)^\lambda, 1-(1-\delta_{UA}^-)^\lambda], \\ [1-(1-\eta_{LA}^-)^\lambda, 1-(1-\eta_{UA}^-)^\lambda] \end{matrix} \right\} \tag{20}$$

Theorem 4

Let A, B, C be three 2HZS, $\lambda, \lambda_1, \lambda_2 > 0$, the following equations are true:

(i) $(A \cdot B)^\lambda = A^\lambda + B^\lambda$ (21)

(ii) $A^{\lambda_1} + A^{\lambda_2} = A^{(\lambda_1+\lambda_2)}$ (22)

Proofs

(i) $(A \cdot B)^\lambda$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-, \\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+, \\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-}} \left(\begin{matrix} [\gamma_{LA}^+\gamma_{LB}^+, \gamma_{UA}^+\gamma_{UB}^+], \\ [\delta_{LA}^+ + \delta_{LB}^+ - \delta_{LA}^+\delta_{LB}^+, \delta_{UA}^+ + \delta_{UB}^+ - \delta_{UA}^+\delta_{UB}^+], \\ [\eta_{LA}^+ + \eta_{LB}^+ - \eta_{LA}^+\eta_{LB}^+, \eta_{UA}^+ + \eta_{UB}^+ - \eta_{UA}^+\eta_{UB}^+], \\ [\gamma_{LA}^-\gamma_{LB}^-, \gamma_{UA}^-\gamma_{UB}^-], \\ [\delta_{LA}^- + \delta_{LB}^- - \delta_{LA}^-\delta_{LB}^-, \delta_{UA}^- + \delta_{UB}^- - \delta_{UA}^-\delta_{UB}^-], \\ [\eta_{LA}^- + \eta_{LB}^- - \eta_{LA}^-\eta_{LB}^-, \eta_{UA}^- + \eta_{UB}^- - \eta_{UA}^-\eta_{UB}^-] \end{matrix} \right)^\lambda$$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-, \\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+, \\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-}} \left(\begin{matrix} [(\gamma_{LA}^+\gamma_{LB}^+)^\lambda, (\gamma_{UA}^+\gamma_{UB}^+)^\lambda], \\ [1-(1-\delta_{LA}^+ - \delta_{LB}^+ - \delta_{LA}^+\delta_{LB}^+)^\lambda, 1-(1-\delta_{UA}^+ - \delta_{UB}^+ - \delta_{UA}^+\delta_{UB}^+)^\lambda], \\ [1-(1-\eta_{LA}^+ - \eta_{LB}^+ - \eta_{LA}^+\eta_{LB}^+)^\lambda, 1-(1-\eta_{UA}^+ - \eta_{UB}^+ - \eta_{UA}^+\eta_{UB}^+)^\lambda], \\ [(\gamma_{LA}^-\gamma_{LB}^-)^\lambda, (\gamma_{UA}^-\gamma_{UB}^-)^\lambda], \\ [1-(1-\delta_{LA}^- - \delta_{LB}^- - \delta_{LA}^-\delta_{LB}^-)^\lambda, 1-(1-\delta_{UA}^- - \delta_{UB}^- - \delta_{UA}^-\delta_{UB}^-)^\lambda], \\ [1-(1-\eta_{LA}^- - \eta_{LB}^- - \eta_{LA}^-\eta_{LB}^-)^\lambda, 1-(1-\eta_{UA}^- - \eta_{UB}^- - \eta_{UA}^-\eta_{UB}^-)^\lambda] \end{matrix} \right)$$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-, \\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+, \\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-}} \left(\begin{matrix} [(\gamma_{LA}^+\gamma_{LB}^+)^\lambda, (\gamma_{UA}^+\gamma_{UB}^+)^\lambda], \\ \begin{bmatrix} 1-(1-\delta_{LA}^+)^\lambda(1-\delta_{LB}^+)^\lambda, \\ 1-(1-\delta_{UA}^+)^\lambda(1-\delta_{UB}^+)^\lambda \end{bmatrix}, \\ \begin{bmatrix} 1-(1-\eta_{LA}^+)^\lambda(1-\eta_{LB}^+)^\lambda, \\ 1-(1-\eta_{UA}^+)^\lambda(1-\eta_{UB}^+)^\lambda \end{bmatrix}, \\ [(\gamma_{LA}^-\gamma_{LB}^-)^\lambda, (\gamma_{UA}^-\gamma_{UB}^-)^\lambda], \\ \begin{bmatrix} 1-(1-\delta_{LA}^-)^\lambda(1-\delta_{LB}^-)^\lambda, \\ 1-(1-\delta_{UA}^-)^\lambda(1-\delta_{UB}^-)^\lambda \end{bmatrix}, \\ [1-(1-\eta_{LA}^-)^\lambda(1-\eta_{LB}^-)^\lambda, 1-(1-\eta_{UA}^-)^\lambda(1-\eta_{UB}^-)^\lambda] \end{matrix} \right)$$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-, \\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+, \\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-}} \begin{pmatrix} \left[(\gamma_{LA}^+)^\lambda(\gamma_{LB}^+)^\lambda, (\gamma_{UA}^+)^\lambda(\gamma_{UB}^+)^\lambda\right], \\ \begin{bmatrix} 1-\left[1-(1-\delta_{LA}^+)^\lambda\right]\left[1-(1-\delta_{LB}^+)^\lambda\right], \\ 1-\left[1-(1-\delta_{UA}^+)^\lambda\right]\left[1-(1-\delta_{UB}^+)^\lambda\right] \end{bmatrix}, \\ \begin{bmatrix} 1-\left[1-(1-\eta_{LA}^+)^\lambda\right]\left[1-(1-\eta_{LB}^+)^\lambda\right], \\ 1-\left[1-(1-\eta_{UA}^+)^\lambda\right]\left[1-(1-\eta_{UB}^+)^\lambda\right] \end{bmatrix}, \\ \left[(\gamma_{LA}^-)^\lambda(\gamma_{LB}^-)^\lambda, (\gamma_{UA}^-)^\lambda(\gamma_{UB}^-)^\lambda\right], \\ \begin{bmatrix} 1-\left[1-(1-\delta_{LA}^-)^\lambda\right]\left[1-(1-\delta_{LB}^-)^\lambda\right], \\ 1-\left[1-(1-\delta_{UA}^-)^\lambda\right]\left[1-(1-\delta_{UB}^-)^\lambda\right] \end{bmatrix}, \\ \begin{bmatrix} 1-\left[1-(1-\eta_{LA}^-)^\lambda\right]\left[1-(1-\eta_{LB}^-)^\lambda\right], \\ 1-\left[1-(1-\eta_{UA}^-)^\lambda\right]\left[1-(1-\eta_{UB}^-)^\lambda\right] \end{bmatrix} \end{pmatrix}$$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-, \\ \gamma_B^+ \in t_B^+,\ \delta_B^+ \in i_B^+, \eta_B^+ \in f_B^+, \\ \gamma_B^- \in t_B^-,\ \delta_B^- \in i_B^-, \eta_B^- \in f_B^-}} \begin{pmatrix} \left[(\gamma_{LA}^+)^\lambda(\gamma_{UA}^+)^\lambda\right], \left[1-(1-\delta_{LA}^+)^\lambda, 1-(1-\delta_{UA}^+)^\lambda\right], \\ \left[1-(1-\eta_{LA}^+)^\lambda, 1-(1-F\eta_{UA}^+)^\lambda\right] \end{pmatrix} \begin{pmatrix} \left[(\gamma_{LA}^-)^\lambda(\gamma_{UA}^-)^\lambda\right], \left[1-(1-\delta_{LA}^-)^\lambda, 1-(1-\delta_{UA}^-)^\lambda\right], \\ \left[1-(1-\eta_{LA}^-)^\lambda, 1-(1-\eta_{UA}^-)^\lambda\right] \end{pmatrix} + \begin{pmatrix} \left[(\gamma_{LB}^+)^\lambda(\gamma_{UB}^+)^\lambda\right], \left[1-(1-\delta_{LA}^+)^\lambda, 1-(1-\delta_{UA}^+)^\lambda\right], \\ \left[1-(1-\eta_{LB}^+)^\lambda, 1-(1-\eta_{UB}^+)^\lambda\right] \end{pmatrix} \begin{pmatrix} \left[(\gamma_{LB}^-)^\lambda(\gamma_{UB}^-)^\lambda\right], \left[1-(1-\delta_{LB}^-)^\lambda, 1-(1-\delta_{UB}^-)^\lambda\right], \\ \left[1-(1-\eta_{LB}^-)^\lambda, 1-(1-\eta_{UB}^-)^\lambda\right] \end{pmatrix}$$

$$= A^\lambda + B^\lambda$$

(ii) $A^{\lambda_1} + A^{\lambda_2}$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-}} \begin{pmatrix} \left[(\gamma_{LA}^+)^{\lambda_1}, (\gamma_{UA}^+)^{\lambda_1}\right], \left[1-(1-\delta_{LA}^+)^{\lambda_1}, 1-(1-\delta_{UA}^+)^{\lambda_1}\right], \\ \left[1-(1-\eta_{LA}^+)^{\lambda_1}, 1-(1-\eta_{UA}^+)^{\lambda_1}\right], \left[(\gamma_{LA}^-)^{\lambda_1}, (\gamma_{UA}^-)^{\lambda_1}\right], \\ \left[1-(1-\delta_{LA}^-)^{\lambda_1}, 1-(1-\delta_{UA}^-)^{\lambda_1}\right], \left[1-(1-\eta_{LA}^-)^{\lambda_1}, 1-(1-\eta_{UA}^-)^{\lambda_1}\right] \end{pmatrix}$$

$$. \begin{pmatrix} \left[(\gamma_{LA}^+)^{\lambda_2}, (\gamma_{UA}^+)^{\lambda_2}\right], \left[1-(1-\delta_{LA}^+)^{\lambda_2}, 1-(1-\delta_{UA}^+)^{\lambda_2}\right], \\ \left[1-(1-\eta_{LA}^+)^{\lambda_2}, 1-(1-\eta_{UA}^+)^{\lambda_2}\right], \left[(\gamma_{LA}^-)^{\lambda_2}, (\gamma_{UA}^-)^{\lambda_2}\right], \\ \left[1-(1-\delta_{LA}^-)^{\lambda_2}, 1-(1-\delta_{UA}^-)^{\lambda_2}\right], \\ \left[1-(1-\eta_{LA}^-)^{\lambda_2}, 1-(1-\eta_{UA}^-)^{\lambda_2}\right] \end{pmatrix}$$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-}} \begin{pmatrix} \left[(\gamma_{LA}^+)^{\lambda_1}(\gamma_{LA}^+)^{\lambda_2}, (\gamma_{UA}^+)^{\lambda_1}(\gamma_{UA}^+)^{\lambda_2}\right], \\ \begin{bmatrix} 1-\left[1-(1-\delta_{LA}^+)^{\lambda_1}\right], \left[1-(1-\delta_{LA}^+)^{\lambda_2}\right], \\ 1-\left[1-(1-\delta_{UA}^+)^{\lambda_1}\right] 1-\left[1-(1-\delta_{UA}^+)^{\lambda_2}\right] \end{bmatrix} \end{pmatrix}$$

$$. \begin{pmatrix} \begin{bmatrix} 1-\left[1-(1-\delta_{LA}^+)^{\lambda_1}\right], \\ \left[1-(1-\delta_{LA}^+)^{\lambda_2}\right], 1-\left[1-(1-\delta_{UA}^+)^{\lambda_1}\right] \\ \left[1-(1-\delta_{UA}^+)^{\lambda_2}\right] \end{bmatrix}, \\ \begin{bmatrix} 1-\left[1-(1-\eta_{LA}^+)^{\lambda_1}\right], \\ \left[1-(1-\eta_{LA}^+)^{\lambda_2}\right], 1-\left[1-(1-\eta_{UA}^+)^{\lambda_1}\right] \\ \left[1-(1-\eta_{UA}^+)^{\lambda_2}\right] \end{bmatrix} \end{pmatrix}$$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-}} \begin{pmatrix} [(\gamma_{LA}^+)^{\lambda_1}(\gamma_{LA}^+)^{\lambda_2}, (\gamma_{UA}^+)^{\lambda_1}(\gamma_{UA}^+)^{\lambda_2}], \\ [1-(1-\delta_{LA}^+)^{\lambda_1}(1-\delta_{LA}^+)^{\lambda_2}, 1-(1-\delta_{UA}^+)^{\lambda_1}(1-\delta_{UA}^+)^{\lambda_2}], \\ [1-(1-\eta_{LA}^+)^{\lambda_1}(1-\eta_{LA}^+)^{\lambda_2}, 1-(1-\eta_{UA}^+)^{\lambda_1}(1-\eta_{UA}^+)^{\lambda_2}], \\ [(\gamma_{LA}^-)^{\lambda_1}(\gamma_{LA}^-)^{\lambda_2}, (\gamma_{UA}^-)^{\lambda_1}(\gamma_{UA}^-)^{\lambda_2}], \\ [1-(1-\delta_{LA}^-)^{\lambda_1}(1-\delta_{LA}^-)^{\lambda_2}, 1-(1-\delta_{UA}^-)^{\lambda_1}(1-\delta_{UA}^-)^{\lambda_2}], \\ [1-(1-\eta_{LA}^-)^{\lambda_1}(1-\eta_{LA}^-)^{\lambda_2}, 1-(1-\eta_{UA}^-)^{\lambda_1}(1-\eta_{UA}^-)^{\lambda_2}] \end{pmatrix}$$

$$= \bigcup_{\substack{\gamma_A^+ \in t_A^+,\ \delta_A^+ \in i_A^+, \eta_A^+ \in f_A^+, \\ \gamma_A^- \in t_A^-,\ \delta_A^- \in i_A^-, \eta_A^- \in f_A^-}} \begin{pmatrix} [(\gamma_{LA}^+)^{(\lambda_1+\lambda_2)}, (\gamma_{UA}^+)^{(\lambda_1+\lambda_2)}], \\ [1-(1-\delta_{LA}^+)^{(\lambda_1+\lambda_2)}, 1-(1-\delta_{UA}^+)^{(\lambda_1+\lambda_2)}], \\ [1-(1-\eta_{LA}^+)^{(\lambda_1+\lambda_2)}, 1-(1-\eta_{UA}^+)^{(\lambda_1+\lambda_2)}], \\ [(\gamma_{LA}^-)^{(\lambda_1+\lambda_2)}, (\gamma_{UA}^-)^{(\lambda_1+\lambda_2)}], \\ [1-(1-\delta_{LA}^-)^{(\lambda_1+\lambda_2)}, 1-(1-\delta_{UA}^-)^{(\lambda_1+\lambda_2)}], \\ [1-(1-\eta_{LA}^-)^{(\lambda_1+\lambda_2)}, 1-(1-\eta_{UA}^-)^{(\lambda_1+\lambda_2)}] \end{pmatrix}$$

$$= A^{(\lambda_1+\lambda_2)}$$

□

6. CONCLUSION

In this paper we have defined some operations for interval-valued bipolar neutrosophic hesitant fuzzy set (IVBNHFS). For convenience, we have used authors' names Hans and Hery Zo Set (2HZS) instead of IVBNHFS. And we have proved some theorems on 2HZS. We hope that the developed theorems and operations will be helpful to decision-making, investment selection, market prediction, teacher selection, school choice, medical diagnosis, pattern recognition, purchasing decision-making, market segment selection, supplier selection, human resource management problems in interval-valued bipolar neutrosophic hesitant fuzzy environment.

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