Stability Issues in Banach Spaces for Functional Equations

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Abstract

Certain nonlinear functional equations are examined for Hyers-Ulam-Rassias stability in this research. Numerous mathematical disciplines have taken such stabilities into consideration. For the second time, fuzzy concepts and their many expansions may be found in almost all areas of mathematics. These intuitionistic fuzzy real Banach spaces are created by combining fuzzy Banach space with intuitionistic fuzzy set concepts. According to Hyers-Ulam-Rassias stability, pexiderized quadratic functional equations defined on such spaces are stable. The issue is approached from a solitary vantage point. On the exact same note, we make use of a generxalized concept of contraction mapping.

Keywords: *Additive Functional Equation, Quadratic Functional Equation, Hyers-Ulam Stability, Quasi-Banach Space, P-Banach Space.*

1. INTRODUCTION

Specifically, in functional analysis, a Banach space is a fully normed vector space. A Banach space is complete when the Cauchy sequence of vectors always converges to the well-defined limit that is inside the space's borders. Vector spaces with metrics enable the calculation of the length and distance between the vectors. Banach spaces are named after Polish mathematician Stefan Banach. Functional analysis relies heavily on Banach spaces. Banach spaces are often used as research areas in a variety of disciplines. [1]

Hyers-Ulam-Rassias stability findings are the stability results found in this research for certain functional equations in Banach spaces. Ulam was the first to mathematically define the stability issue that we are looking into in this study. Hyers and Rassias only addressed a portion of the issue by broadening the solution's applicability. Concerns about stability, such as the recent financial crisis, are all too common in today's world. In mathematics, it may be used in a broad variety of areas, such as differential equations, functional equations, symmetry, and many more things. There are a number of alternate extended versions of this. [2]

Convergence sequence in quasi-norm space turns out to have a number of intriguing properties. We show that in this specific case, there is a complete quasi-normed space or pseudo Banach space. The converse is not true for any quasi-Banach space, as we show. [3]

In 1940, Ulam stumbled upon his first problem: a lack of steadiness. Since then, a number of mathematicians have focused on the problem of stability. Zadeh was the first to propose fuzzy sets back in 1965. Because of her classical upbringing, Zadeh has made it a point to explore the gray areas between academic ideas. For the purpose of better understanding the problems with functional equation stability. [4]

2. LITERATURE REVIEW

According to Krzysztof Ciepliski (2016), two general functional equations with many variables in 2-Banach spaces are Ulam stable. We will examine the stability of a few well-known equations that are special instances of the ones being investigated as a logical extension of our main results. This article extends previous research on the Ulam stability of functional equations in 2-Banach spaces.[6]

Theodore Nuino (2015) Functional equations are a relatively new area of mathematics. The current concept of function, which came into being about the same time as functional equations, is where their roots lie. Functional equations have been studied for almost 260 years, although the field has grown significantly in the preceding 70 years. Quadratic functional equations, bilinear forms similar to the quadratic equation, and other generalizations have been addressed by many writers in diverse contexts.[7]

Ismail Bodghi (2014) As a result of this work, we may define the generalized multi-quadratic functional equation as a single equation with multiple distinct multi-variable mappings that are quadratic in each variable. A fixed-point theorem is used to demonstrate Hyers–Ulam stability for generalized multi-quadratic functional equations. The examples and conclusions presented here are based on well-known findings in the field of stability.[8]

Nazek Alessa, K. Tamilvanan, G. Balasubramanian, and K. Loganathan are among the researchers (2017) The authors present a finite-variable quadratic functional problem and subsequently prove its solution. According to the authors, there are two methods to determine if Hyers-Ulam stability exists in Random Normed space (RN-space).[9]

Preliminaries

The multi-Banach space was first investigated by Dales and Polyakov. Theory of multi-Banach spaces is similar to the operator sequence space and has some connections with operator spaces and Banach spaces. In 2007, H. G. Dales and M. S. Moslehian first proved the stability of mappings on multi-normed spaces and also gave some examples on multi-normed spaces. The asymptotic aspects of the quadratic functional equations in multi-normed spaces were investigated by M. S. Moslehian, K. Nikodem, and D. Popar in 2009. In last two decades, the stability of functional equations on multi-normed spaces was proved by many mathematicians.

Now, we adopt some usual terminology, notion and convention of the theory of multi-Banach spaces from and be a complex normed space, and let k·Let (E, || E consisting⊕…⊕N. We denote by Ek the linear space E∈ E. The linear∈of k-tuples (x1,…, xk), where x1,…, xk operations on Ek are defined coordinate-wise. The zero element of either E or Ek is denoted by 0. We denote by Nk the set and by Sk the group of permutations on k symbols.

3. STABILITY PROBLEM IN BANACH SPACE

Ulam was the first to raise the issue of stability in his writings in 1940. Hyers found a solution to Ulam's conundrum in 1941. Aoki and Th extended this finding to additive mappings. M. Rassias considers an unbounded Cauchy difference for linear mappings.

Theorem 1.1 Th. M. Rassias. Let f : E → E be a mapping from a normed vector space E into aBanach space E subject to the inequality:

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all x, y ∈ E, where and p are constants with > 0 and 0 ≤ p < 1. Then, the limit L x limn → ∞ 1/2nf 2nx exists for all x ∈ E and L: E → E is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p$$

for all x ∈ E. Also, if for each x ∈ E the function f tx is continuous in t ∈ R, then L is linear In 1990, Th. M. Rassias 5 during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for p ≥ 1. In 1991, Gajda 6 gave an affirmative solution to this question for p > 1. It was shown by Gajda 6, aswell as by Th. M. Rassias and Semrl ˇ 7, that one cannot prove a Th. M. Rassias type theoremwhen p 1. Gavrut ˘ ¸a 8 proved that the function f x x ln |x|, if x / 0 and f 0 satisfies 1.1 with p 1 but

$$\sup_{x\neq 0}\frac{|f(x) - A(x)|}{|x|} \geq \sup_{n\in\mathbb{N}}\frac{|n \ln n - A(n)|}{n} = \sup_{n\in\mathbb{N}}|\ln n - A(1)| = \infty$$

for any additive function A : R → R. J. M. Rassias 9 replaced the factor xp yp by xp1 yp2 for p1, p2 ∈ R with p1 p2 / 1 see also 10, 11and has obtained the following theorem

Theorem 1.2. Let X be a real normed linear space and Y a real complete normed linear space. Assume that f : X → Y is an approximately additive mapping for which there exist constants θ ≥ 0 and p p1 p2 / 1 such that f satisfies the inequality:

$$\|f(x+y)-f(x)-f(y)\| \le \theta\|x\|^{p_1}\|y\|^{p_2}$$

for all x, y ∈ X. Then, there exists a unique additive mapping L : X → Y satisfying

$$\|f(x)-L(x)\| \le \frac{\theta}{|2^p-2|}\|x\|^p$$

for all x ∈ X. If, in addition, f:X → Y is a mapping such that the transformation t → f tx is continuous in t ∈ R for each fixed x ∈ X, then L is an R-linear mapping.

Theorem 1.3. Let E1 and E2 be two Banach spaces, and let f : E1 → E2 be a mapping such that f tx is continuous in t for each fixed x. Assume that there exist θ ≥ 0 and p ∈ 0, 1such that

$$\|f(x+y)-f(x)-f(y)\| \le \theta(\|x\|^p+\|y\|^p)$$

for all x, y ∈ X. Let k be a positive integer k > 2. Then, there exists a unique linear mapping T : E1 → E2 such that

$$\|f(x)-T(x)\| \le \frac{k\theta}{k-k^p}\|x\|^p s(k,p)$$

for all x ∈ X, where

$$s(k,p)=1+\frac{1}{k}\sum_{m=2}^{k-1}m^p.$$

Th. M. Rassias Problem What is the best possible value of k in Theorem 1.3? Gavrut ˘ ¸a et al. have given a generalization of 13 and have answered to Th. M. Rassias problem 44. In 45, J. M. Rassias et al. have investigated the generalized Ulam-Hyers "productsum" stability of functional equations and have obtained the following theorem

Theorem 1.4 see 45. Let f : E → F be a mapping which satisfies the inequality

$$\left\|f(mx+y)+f(mx-y)-2f(x+y)-2f(x-y)-2\left(m^2-2\right)f(x)+2f(y)\right\|_F \le \epsilon\left(\|x\|_E^p\|y\|_E^p+\|x\|_E^{2p}+\|y\|_E^{2p}\right)$$

for all x, y ∈ E with x ⊥ y, where and p are constants with , p > 0 and either m > 1, p < 1 or m < 1, p > 1 with m / 0, m / ± 1, m / √ ±2, and −1 / |m| p−1 < 1. Then, the limit limn → ∞m−2nf mnx exists for all x ∈ E and Q : E → F is the unique orthogonally Euler-Lagrange quadratic mapping such that

$$\|f(x)-Q(x)\|_F \le \frac{\epsilon}{2|m^2-m^{2p}|}\|x\|_E^{2p}$$

The pair X, N is called a fuzzy normed linear space. The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in 49–51. Let X, N be a fuzzy normed space and let {xn} be a sequence in X. Then, {xn} is said to be convergent if there exists x ∈ X such that limn → ∞N xn − x, t 1 for all t > 0. In that case, x is called the limit of the sequence {xn} and we denote it by limn → ∞xn x. A sequence {xn} in a fuzzy normed space X, Nis called Cauchy if, for each > 0 and δ > 0, one can find some n0 such that

$$N(x_m-x_n,\delta)>1-\epsilon$$

for all n, m ≥ n0. It is known that every convergent sequence in a fuzzy normed space is Cauchy. If, in a fuzzy-normed space, each Cauchy sequence is convergent, then the fuzzy-norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space. Stability of Cauchy, Jensen, quadratic, and cubic function equation in fuzzy normed spaces have first been investigated in 50–53. In this paper, we give a generalization of the results from 13 and pose two open problems in fuzzy Banach space. For convenience, we use the following abbreviation for a given mapping f:

$$Df(x,y) =: f(x+y) - f(x) - f(y)$$

4. FUNCTIONAL EQUATION

An equation of the form f(x,y,...)=0, where f contains a finite number of independent variables, known functions, and unknown functions which are to be solved for. Many properties of functions can be determined by studying the types of functional equations they satisfy. For example, the gamma function Gamma(z) satisfies the functional equations.

$$\Gamma(1+z) = z\,\Gamma(z)$$
$$\Gamma(1-z) = -z\,\Gamma(-z)$$

The following functional equations hold

$$f(x) = f(x+1) + f\left(x^2+x+1\right)$$
$$l(x) = l(2x+1) + l(2x)$$
$$\tau(x) = \tau(x+1) + \tau\left(x^2+x-1\right)$$
$$\sigma(x) = \sigma\left(\sqrt{x^2+1}\right) + \sigma\left(x\sqrt{x^2+1}\left(\sqrt{x^2+1}+\sqrt{x^2+2}\right)\right)$$
$$\rho(x) = \rho\left(\sqrt{x^2+1}\right) + \rho\left(x\sqrt{x^2+1}\left(x+\sqrt{x^2-1}\right)\right)$$
$$\rho(x) = \rho\left(\frac{x^2}{(x-1)\sqrt{x^2-1}+\sqrt{2x+1}}\right) - \rho\left(\frac{x}{x-1}\right),$$

Where

$$f(x) = \tan^{-1}\left(\frac{1}{x}\right)$$
$$l(x) = \ln\left(1+\frac{1}{x}\right)$$
$$\tau(x) = \tanh^{-1}\left(\frac{1}{x}\right) = \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right)$$
$$\sigma(x) = \sinh^{-1}\left(\frac{1}{x}\right)$$
$$\rho(x) = \sin^{-1}\left(\frac{1}{x}\right)$$

5. TYPES OF BANACH SPACES

Banach spaces are one of the most important study subjects in functional analysis, a branch of mathematics. Most spaces that arise in practise in other areas of mathematics analysis turn out to be Banach spaces as well.

Asplund Spaces

An Asplund space or a strong differentiability space is a well-behaved Banach space in mathematics, specifically in functional analysis. After studying Lipschitz functions on Banach spaces and their Fréchet differentiability, mathematician Edgar Asplund devised the Asplund space theory in 1968.

The Hardy Spaces

Certain holomorphic function spaces on the unit disc or upper half plane are called Hardy spaces (or Hardy classes) in complex analysis. Frigyes Riesz (Riesz 1923) first used them and called them after G. H. Hardy as a result of the paper (Hardy 1915). Hardy spaces are specific distribution spaces on the real line in real analysis.

The Space of Functions of Bounded Mean Oscillation

A function with bounded mean oscillation (BMO) is a real-valued function whose mean oscillation is bound in harmonic analysis in mathematics (finite). It is also known as the John–Nirenberg space after Fritz John and Louis Nirenberg, who first introduced and studied it. The space of bounded mean oscillation functions (BMO) is a function space that, in some precise sense, plays the same role in the theory of Hardy spaces Hp as the space L of essentially bounded functions plays in the theory of Lp-spaces:

The Space of Functions of Bounded Variation

Boundary-variation functions can and often are used to describe generalised solutions to nonlinear mathematical, physical, and engineering problems that involve functionals, ordinary differential equations, and partial differential equations. This makes them extremely useful in these fields.

Sobolev Spaces

Sergei Sobolev, a Russian mathematician, gave his name to Sobolev spaces. Due to the fact that weak solutions to several significant partial differential equations can exist in appropriate Sobolev spaces even when no strong solutions are present for continuous functions with classical derivatives, their significance is derived from this fact.

The Birnbaum–Orlicz Spaces

A generalisation of the Lp spaces, the Orlicz space is a form of function space in mathematical analysis, particularly in real and harmonic analysis. They are Banach spaces, just like the Lp spaces. They are named for Wadysaw Orlicz, who initially defined them in 1932 and is credited with giving them their current name.

Hölder Space

the property of being Hölder continuous means that a function f in Euclidean space with real or complex values meets a Hölder condition.

Lorentz Space

Generalizations of the more familiar displaystyle lpLp spherical

Quasi Banach Space

Functional analysis is a relatively new scientific discipline. For example, in physics, engineering, medicine, agro-industry and ecology, it provides a powerful instrument for finding solutions to problems that arise in pure and applied social sciences. When doing functional analysis, it's critical to understand what Banach space is all about. Introducing this concept in 1922, Polish mathematician Stefan Banach garnered a lot of attention and references in the literature.

The purpose of this paper is to introduce the notion of a quasi-Banach space for sequence space p λ where 1.< p <0 This paper is organised as follows. devotes an introduction of the sequence space , p λ 1.< p <0 We give the definition of sequence space , p λ where 1,< p <0 and we show that it is not normed space.we prove that, this space is a quasi-normed space and we give some interesting results concerning this notion. In the last section, we study the convergence and completeness sequence in a quasi-normed space , p 1< p < 0 λ in order to show that it is a quasi-Banach space.

P -Banach Space

Many topics in analysis involve the use of function spaces, particularly Lp spaces. Due to their incomplete but beneficial generalisation of the fundamental L space of square integrable functions, Lp spaces have a unique significance. Basic structural facts concerning Lp spaces are covered in detail in this chapter. The study of their linear functionals, in particular, is best articulated in the more general framework of Banach spaces in this part of the theory.

Using a more abstract viewpoint has the side benefit of leading us to the unexpected finding of an additive measure on all sets that is compatible with the Lebesgue measure. Since it appears in the description of functions that are integrable in the Lebesgue sense, the space L comes first in the logical order of simplicity. The L space of bounded functions, whose supremum norm comes from the more familiar space of continuous functions, is linked to it via duality. The L space, whose roots are connected to fundamental Fourier analysis concerns, is also of interest.

In this sense, the intermediary Lp spaces are a fabrication, albeit a very brilliant and fortunate one. The following and succeeding chapters' findings will demonstrate this.

Non Linear L –Fuzzy Banach Spaces

In many L fuzzy topology situations, it is necessary to gain a thorough understanding of a L FNS. Saadati and Park introduced and investigated the intuitionistic FNS theory. As time went on, researchers such as Deschrijver et al. and Saadati elaborated on intuitionistic FNS's and looked into and generalised the idea of L FNS's. In this article, fuzzy Banach space (abbreviated FBS) and fuzzy normed space (abbreviated FNS) are used interchangeably. We're reminded of a recent idea that's appeared in the literature.

6. CONCLUSION

To explore the Hyers-Ulam stability of the mixed kind of functional equation, we used the Hyers direct technique described in this paper. We've also found a generic solution to the problem. Ulam-Hyers stability was examined for the mixed situations in intuitionistic banach space, which was the major goal of this research.

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