THE RELATIONSHIP BETWEEN QUADRATIC AVERAGE, ARBITRARY LEVEL AVERAGE AND ARITHMETIC AVERAGE

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ABSTRACT

Functional equalities is one of the oldest topics of mathematical analysis, and it is attentioned of reseachers . In 1964, M. Kuczma, A survay of the theory of functional equation (see, [1]) . In 1966, T. Aczel , Lectures on functional equations and their applications (see, [2]) . In 2000, Christopher G. Mall. , Functional eqautions and how to solve them (see, [3]) . In 2002, B. J. Venkatachala , Functional eqautions – A problem solving aproach (see, [4]. The theory of functional equalities was born very early. Functiona equalities is presented in almost every field and have many applications in life and engineering. There is a lot of Functional equalities. In this paper, we establish the relationship between quadratic average, arbitrary level average, and arithmetic average.To solve some mathematical problems via functional equalities, we use method of substitution.

Keyword: *Functional equalities, quadratic average, arbitrary level average, and and arithmetic average.*

1. PRELIMINARIES

In this paper, we derive some mathematical problems via functional equalities
arbitrary level average

$$\frac{x+y}{2}; x, y \in \mathbb{R};$$

$$\sqrt{\frac{x^2+y^2}{2}}; x, y \in \mathbb{R}^+;$$

$$\sqrt[t]{\frac{x^t+y^t}{2}}; x, y \in \mathbb{R}^+, t > 1;$$

and
arithmetic average

$$\frac{f(x)+f(y)}{2}$$

In reality, to solve some mathematical problems, we often use method of substitution
+) Example, let $x = X$ such that $f(X)$ appears much in the equation.

+) Let $x = X, y = Y$ interchange to refer $f(a)$ and $f(b)$.
+) Let $f(0) = b, f(1) = b$,
+) If f is surjection, exist $a: f(a) = 0$. Choice x, y to destroy $f(g(x,y))$ in the equation. The function has x, we show that it is injective or surjection.
+) To occur $f(x)$.
+) $f(x) = f(y)$ for all x, y ∈ A. Hence $f(x) = const$ for all $x \in A$.

2. THE RELATIONSHIP BETWEEN QUADRATIC AVERAGE, ABRITRARY LEVEL AVERAGE, AND ARITHMETIC AVERAGE

In section, we consider theralatìonship between quadratic average, arbitrary level average, and arithmetic average.

2.1 Problem 1. Determine all functions $f(x)$ continuous on $\mathbb{R}$ such that:

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}, \forall x, y \in \mathbb{R}. \quad (2.1.1)$$

Solution. Let $f(0) = b$ and $f(x) = b + g(x)$. Then $g(0) = 0$. Replacing (2.1.1), we get

$$b + g\left(\frac{x+y}{2}\right) = \frac{2b + g(x) + g(y)}{2}, \forall x, y \in \mathbb{R},$$

$$g\left(\frac{x+y}{2}\right) = \frac{g(x)+g(y)}{2}, \forall x, y \in \mathbb{R}. \quad (2.1.2)$$

with $g(0) = 0$.
Let $y = 0$. Then (2.1.2), we have

$$g\left(\frac{x}{2}\right) = \frac{g(x)}{2},$$

Or

$$g\left(\frac{x+y}{2}\right) = \frac{g(x)+g(y)}{2}, \forall x, y \in \mathbb{R}.$$

Replacing (2.1.2), we have

$$g\left(\frac{x+y}{2}\right) = \frac{g(x)+g(y)}{2}, \forall x, y \in \mathbb{R}.$$

$$g(x+y) = g(x) + g(y), \forall x, y \in \mathbb{R}. \quad (2.1.3)$$

Since $g(x)$ is continuous on $\mathbb{R}$. Then (2.1.3) is Cauchy function. Then $g(x) = ax$.
We get $f(x) = ax + b, a, b \in \mathbb{R}$.
We can check directly $f(x) = ax + b$ satisfies (2.1.1).
Hence,

$$f(x) = ax + b, with\ for\ arbitrary\ \ a, b \in \mathbb{R}.$$

2.2 Problem 2. Determine all functions $f(x)$ continuous on $\mathbb{R}$ such that

$$f\left(\sqrt{\frac{x^2+y^2}{2}}\right) = \frac{f(x)+f(y)}{2}, \forall x, y \in \mathbb{R}. \quad (2.2.1)$$

Solution. By assumption, we have $f(x) = f(|x|), \forall x \in \mathbb{R}$.

Let $|x| = \sqrt{X}, |y| = \sqrt{Y}\ (X, Y \geq 0)$, we have

$$(2.2.1) \Leftrightarrow f\left(\sqrt{\frac{X+Y}{2}}\right) = \frac{f\left(\sqrt{X}\right)+f\left(\sqrt{Y}\right)}{2}, \forall X,Y \geq 0.$$

Let $f\left(\sqrt{X}\right) = g\left(X\right), X \geq 0,$ we have

$$g\left(\frac{X+Y}{2}\right) = \frac{g\left(X\right)+g\left(Y\right)}{2}, \forall X,Y \geq 0.$$

By Problem 2.1, we have

$$g\left(X\right) = aX + b.$$

Hence,

$$f\left(\sqrt{X}\right) = aX + b, X \geq 0,$$

and

$$f\left(X\right) = aX^2 + b, X \geq 0.$$

Thus,

$$f\left(x\right) = f\left(|x|\right) = \text{ax}^2 + b, \forall a,b \in \mathbb{R}.$$

We can check directly $f\left(x\right) = f\left(|x|\right) = \text{ax}^2 + b, \forall a,b \in \mathbb{R}.$ satisfies problem.
Hence,

$$f\left(x\right) = \text{ax}^2 + b, \forall a,b \in \mathbb{R}.$$

2.3 Problem 3. Determine all functions $f(x)$ continuous on $\mathbb{R}$ such that:

$$f\left(\sqrt[k]{\frac{x^k+y^k}{2}}\right) = \frac{f\left(x\right)+f\left(y\right)}{2}, \forall x,y \in \mathbb{R}, k = 1,2.... \quad (2.3.1)$$

Solution. By assumption, we have $f\left(x\right) = f\left(|x|\right), \forall x \in \mathbb{R}.$

Let $|x| = \sqrt[k]{X}, |y| = \sqrt[k]{X} \; \left(X,Y \geq 0\right),$ we have

$$(2.3.1) \Leftrightarrow f\left(\sqrt{\frac{X+Y}{2}}\right) = \frac{f\left(\sqrt[k]{X}\right)+f\left(\sqrt[k]{Y}\right)}{2}, \forall X,Y \geq 0.$$

Let $f\left(\sqrt[k]{X}\right) = g\left(X\right), X \geq 0,$ we have

$$g\left(\frac{X+Y}{2}\right) = \frac{g\left(X\right)+g\left(Y\right)}{2}, \forall X,Y \geq 0.$$

By Problem 2.1, we have $g\left(X\right) = aX + b.$

Hence, $f\left(\sqrt[k]{X}\right) = aX + b, X \geq 0,$ and $f\left(X\right) = aX^k + b, X \geq 0.$

Thus $f\left(x\right) = f\left(|x|\right) = \text{ax}^k + b, \forall a,b \in \mathbb{R}.$ We can check directly $f(x)$ satisfies problem.
Hence,

$$f\left(x\right) = \text{ax}^k + b, \forall a,b \in \mathbb{R}.$$

3. CONCLUSIONS

Functional equalities are very difficult and are frequently exploited in regional, national and international olympiads. The development used to application in several areas - not only in mathematics but also in other disciplines. Functional equalities are applied computers, economics, polynomials, engineering, information theory, reproducing scoring system, taxation, etc. In this paper, we establish the relationship between quadratic average, arbitrary level average, and arithmetic average.

4. REFERENCES

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