A BRIEF VIEW ABOUT COMPLEX ANALYSIS IN MATHEMATICS

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ABSTRACT;- Complex analysis, in particular the theory of conformal mappings, has many physical applications and is also used throughout analytic number theory. A complex function is a function from complex numbers to complex numbers. In other words, it is a function that has a subset of the complex numbers as a domain and the complex numbers as a codomain. Complex functions are generally supposed to have a domain that contains a nonempty open subset of the complex plane. Complex Analysis, is divided into four blocks, which are further subdivided into fourteen units. This book provides a basic understanding of the subject and helps to grasp its fundamentals. In a nutshell, it explains various aspects, such as complex numbers, geometric representation of a complex number, the spherical representation and stereographic projection, complex analytic function, complex differentiation, the Cauchy-Riemann equations, orthogonal trajectories and harmonic functions, Milne-Thomson method, power series, radius of convergence, Complex analysis is full of amazingly beautiful theorems that have no analogue in standard calculus. For instance, if you know the value of a real-valued function on some circle in the plane, that is not enough to determine its value at any point in the interior. Yet, for complex differentiable functions, this is sufficient, thanks to the delightful.

KEYWORDS;- SPHERICAL REPRESENTATION AND STEREOGRAPHIC PROJECTION, COMPLEX ANALYTIC FUNCTION, COMPLEX DIFFERENTIATION, GENERATING FUNCTIONS, FIBONACCI NUMBERS, CONFORMAL MAPPINGS, BILINEAR TRANSFORMATIONS, COMPLEX INTEGRATION, CAUCHY'S THEOREM

INTRODUCTION

Complex analysis, traditionally known as the theory of functions of a complex variable, is the branch of mathematical analysis that investigates functions of complex numbers. It is useful in many branches of mathematics, including algebraic geometry, number theory, analytic combinatorics, applied mathematics; as well as in physics, including the branches of hydrodynamics, thermodynamics, and particularly quantum mechanics. By extension, use of complex analysis also has applications in engineering fields, such as nuclear, aerospace, mechanical and electrical engineering.

As a differentiable function of a complex variable is equal to its Taylor series, i.e., it is analytic, also the complex analysis is particularly concerned with analytic functions of a complex variable, i.e., holomorphic functions. Fundamentally, 'Complex Analysis' is one of the classical branches in mathematics, with roots in the 18th century and just prior. Important mathematicians associated with complex numbers include Euler, Gauss, Riemann, Cauchy, Weierstrass, and many more in the 20th century.

Complex analysis, in particular the theory of conformal mappings, has many physical applications and is also used throughout analytic number theory. A complex function is a function from complex numbers to complex numbers. In other words, it is a function that has a subset of the complex numbers as a domain and the complex numbers as a codomain. Complex functions are generally supposed to have a domain that contains a nonempty open subset of the complex plane.

One of the central tools in complex analysis is the line integral. The line integral around a closed path of a function that is holomorphic everywhere inside the area bounded by the closed path is always zero, as is stated by the Cauchy integral theorem.

The values of such a holomorphic function inside a disk can be computed by a path integral on the disk's boundary. The study of complex analysis is important for students in engineering and the physical sciences and is a central subject in mathematics. In addition, complex analysis provides powerful tools for solving problems that are either very difficult or virtually impossible to solve in any other way.

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Abel's limit theorem, generating functions, Fibonacci numbers, conformal mappings, bilinear transformations, complex integration, Cauchy's theorem for a rectangle and for a disk, Cauchy's integral formula, higher derivatives in complex integration,

Taylor's theorem, zeros and poles, local mapping theorem, the maximum principle, Schwarz's lemma, Morera's theorem, Cauchy's estimate, Liouville's theorem, fundamental theorem of algebra, series expansions, Taylor's series, Laurent series, Laurent's theorem, singularities, Cauchy's residue theorem, the argument principle, Rouche's theorem, and evaluation of definite integrals.

Complex analysis (Winter and Spring 2021) is the study of calculus with complex numbers in place of real numbers. At a first glance, it may appear that there should not be much difference between calculus with real numbers and calculus with complex numbers, but nothing could be further from the truth!

Complex analysis is full of amazingly beautiful theorems that have no analogue in standard calculus. For instance, if you know the value of a real-valued function on some circle in the plane, that is not enough to determine its value at any point in the interior. Yet, for complex differentiable functions, this is sufficient, thanks to the delightful.

formula $f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0}\,dz$, known as the Cauchy Integral Formula. Using this formula and a bit more, one can turn many integrals into finite computations, which even allows for easy computations of some real integrals, such as $\int_{-\infty}^{\infty} \frac{1}{1+x^8}\,dx$.

Complex analysis can also be used to clear up some mysteries about functions of a real variable. For instance, consider the series $1 + x + x^2 + x^3 + \cdots$. This series converges when $|x| < 1$, and it converges to $\frac{1}{1-x}$. That makes sense: when $x = 1$, the function $\frac{1}{1-x}$ is not defined, so it is reasonable that the series will also encounter problems there. But what about the series $1 - x^2 + x^4 - x^6 + \cdots$? This series also converges when $|x| < 1$, but this time it converges to $\frac{1}{1+x^2}$. But at $x = \pm 1$, $\frac{1}{1+x^2}$ is a perfectly well-behaved function. What is going on? Well, complex numbers come to the rescue, and remind us that this function does not behave so well at $x = i$, and this can be used to explain the failure of convergence.

In the land of real functions, we are familiar with functions like $f(x) = \sin(x)$, which only take on a narrow range of values (in this case, $-1 \leq \sin(x) \leq 1$). It turns out that this is purely a phenomenon of real functions; all complex functions that are defined everywhere (except for constant functions) must take on a wide range of values. There are various theorems along these lines, starting from Liouville's Theorem and progressing to the more demanding Picard's Theorem. The Fundamental Theorem of Algebra, which says that every nonconstant polynomial over the complex numbers has a complex root, is a very easy consequence of Liouville's Theorem.

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Murray R. Spiegel described complex analysis as "one of the most beautiful as well as useful branches of Mathematics".

Complex analysis is particularly concerned with the analytic functions of complex variables (or, more generally, meromorphic functions). Because the separate real and imaginary parts of any analytic function must satisfy Laplace's equation, complex analysis is widely applicable to two-dimensional problems in physics.

History

The Mandelbrot set, a fractal.

Complex analysis is one of the classical branches in mathematics with roots in the 19th century and just prior. Important names are Euler, Gauss, Riemann, Cauchy, Weierstrass, and many more in the 20th century. Complex analysis, in particular the theory of conformal mappings, has many physical applications and is also used throughout analytic number theory. In modern times, it has become very popular through a new boost from complex dynamics and the pictures of fractals produced by iterating holomorphic functions. Another important application of complex analysis is in string theory which studies conformal invariants in quantum field theory.

Complex functions

A complex function is one in which the independent variable and the dependent variable are both complex numbers. More precisely, a complex function is a function whose domain and range are subsets of the complex plane.

For any complex function, both the independent variable and the dependent variable may be separated into real and imaginary parts:

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where *[Math Processing Error]*, are real-valued functions.

In other words, the components of the function f(z),

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can be interpreted as real-valued functions of the two real variables, x and y.

The basic concepts of complex analysis are often introduced by extending the elementary real functions (e.g., exponentials, logarithms, and trigonometric functions) into the complex domain.

Holomorphic functions

Holomorphic functions are complex functions defined on an open subset of the complex plane that are differentiable. Complex differentiability has much stronger consequences than usual (real) differentiability. For instance, holomorphic functions are infinitely differentiable, whereas some real differentiable functions are not. Most elementary functions, including the exponential function, the trigonometric functions, and all polynomial functions, are holomorphic. See also: analytic function, holomorphic sheaf and vector bundles.

Major results

One central tool in complex analysis is the line integral. The integral around a closed path of a function that is holomorphic everywhere inside the area bounded by the closed path is always zero; this is the Cauchy integral theorem. The values of a holomorphic function inside a disk can be computed by a certain path integral on the disk's boundary (Cauchy's integral formula). Path integrals in the complex plane are often used to determine complicated real integrals, and here the theory of residues among others is useful (see methods of contour integration). If a function has a pole or singularity at some point, that is, at that point where its values "blow up" and have no finite boundary, then one can compute the function's residue at that pole. These residues can be used to compute path integrals involving the function; this is the content of the powerful residue theorem.

The remarkable behavior of holomorphic functions near essential singularities is described by Picard's Theorem. Functions that have only poles but no essential singularities are called meromorphic. Laurent series are similar to Taylor series but can be used to study the behavior of functions near singularities.

A bounded function that is holomorphic in the entire complex plane must be constant; this is Liouville's theorem. It can be used to provide a natural and short proof for the fundamental theorem of algebra which states that the field of complex numbers is algebraically closed.

If a function is holomorphic throughout a simply connected domain then its values are fully determined by its values on any smaller subdomain. The function on the larger domain is said to be analytically continued from its values on the smaller domain. This allows the extension of the definition of functions, such as the Riemann zeta function, which are initially defined in terms of infinite sums that converge only on limited domains to almost the entire complex plane. Sometimes, as in the case of the natural logarithm, it is impossible to analytically continue a holomorphic function to a non-simply connected domain in the complex plane but it is possible to extend it to a holomorphic function on a closely related surface known as a Riemann surface.

Complex analysis is the study of complex numbers together with their derivatives, manipulation, and other properties. Complex analysis is an extremely powerful tool with an unexpectedly large number of practical applications to the solution of physical problems. Contour integration, for example, provides a method of computing difficult integrals by investigating the singularities of the function in regions of the complex plane near and between the limits of integration.

The key result in complex analysis is the Cauchy integral theorem, which is the reason that single-variable complex analysis has so many nice results. A single example of the unexpected power of complex analysis is Picard's great theorem, which states that an analytic function assumes every complex number, with possibly one exception, infinitely often in any neighborhood of an essential singularity!

A fundamental result of complex analysis is the Cauchy-Riemann equations, which give the conditions a function must satisfy in order for a complex generalization of the derivative, the so-called complex derivative, to exist. When the complex derivative is defined "everywhere," the function is said to be analytic.

Complex analysis is known as one of the classical branches of mathematics and analyses complex numbers concurrently with their functions, limits, derivatives, manipulation, and other mathematical properties. Complex analysis is a potent tool with an abruptly immense number of practical applications to solve physical problems. Let’s understand various components of complex analysis one by one here.

Complex numbers

A number of the form x + iy where x, y are real numbers and $i^2 = -1$ is called a complex number.

In other words, z = x + iy is the complex number such that the real part of z is x and is denoted by Re(z), whereas the imaginary part of z is iy and is denoted by I(z).

Modulus and Argument of a Complex Number

The modulus of a complex number z = x + iy is the real number $\sqrt{(x^2 + y^2)}$ and is denoted by |z|.

The amplitude or argument of a complex number z = x + iy is given by:

$\arg(z) = \theta = \tan^{-1}(y/x)$, where $x, y \neq 0$.

Also, the arg(z) is called the principal argument when it satisfies the inequality $-\pi < \theta \leq \pi$, and it is denoted by Arg(z).

Complex Functions

In complex analysis, a complex function is a function defined from complex numbers to complex numbers. Alternatively, it is a function that includes a subset of the complex numbers as a domain and the complex numbers as a codomain. Mathematically, we can represent the definition of complex functions as given below:

A function $f : C \to C$ is called a complex function that can be written as

w = f(z), where $z \in C$ and $w \in Z$.

Also, z = x + iy and w = u + iv such that u = u(x, y) and v = v(x, y). That means u and v are functions of x and y.

Limits of Complex Functions

Let w = f(z) be any function of z defined in a bounded closed domain D. Then the limit of f(z) as z approaches z_0 is denoted by “l”, and is written as

$$\lim_{z \to z_0} f(z) = l$$

, i.e., for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(z) - l| < \epsilon$ whenever $|z - z_0| < \delta$ where ϵ and δ are arbitary small positive real numbers. Here, l is the simultaneous limit of f(z) as $z \to z_0$.

Continuity of Complex Functions

Let's understand what is the continuity of complex functions in complex analysis.

A complex function w = f(z) defined in the bounded closed domain D, is said to be continuous at a point Z0, if f(z0) is defined

$$\lim_{z \to z_0} f(z) = f(z_0)$$

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Complex Differentiation

Some of the standard results of complex differentiation are listed below:

- dc/dz = 0; where c is a complex constant
- d/dz (f ± g) = (df/dz) ± (dg/dz)
- d/dz [c.f(z)] = c . (df/dz)
- d/dz $z^n = nz^{n-1}$
- d/dz (f.g) = f (dg/dz) + g (df/dz)
- d/dz (f/g) = [g (df/dz) – f (dg/dz)]/ g^2

All these formulas are used to solve various problems in complex analysis.

Analytic Functions

A function f(z) is said to be analytic at a point z_0 if f is differentiable not only at z, but also at every point in some neighbourhood of z_0. Analytic functions are also called regular, holomorphic, or monogenic functions.

Harmonic Function

A function u(x, y) is said to be a harmonic function if it satisfies the Laplace equation. Also, the real and imaginary parts of an analytic function are harmonic functions.

Complex Integration

Suppose f(z) be a function of complex variable defined in a domain D and "c" be the closed curve in the domain D.

Let f(z) = u(x, y) + i v(x, y)

Here, z = x + iy

(or)

$f(z) = u + iv$ and $dz = dx + i\,dy$

$\int_c f(z)\,dz = \int_c (u + iv)\,dz$

$= \int_c (u + iv)(dx + idy)$

$= \int_c (udx - vdy) + i \int_c (udy + vdx)$

Here, $\int_c f(z)\,dz$ is known as the contour integral.

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