A STUDY OF DOMINATION NUMBER OF FEW GRAPH THEORY

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ABSTRACT;- An elementary contraction of a graph G is a transformation of G to a new graph G_1, such that two adjacent vertices u and υ of G are replaced by a new vertex w in G_1 and w is adjacent in G_1 to all vertices to which either u or υ is adjacent in G. A graph G^* is said to be a contraction of G if G^* can be obtained from G by a sequence of elementary contractions. The following is another characterization of a planar graph due to the German mathematician K. Wagner in 1937. A graph is planar if and only if it is not contractible to K_5 or $K_{3,3}$. A graph G is said to be planar if it can be represented on a plane in such a fashion that the vertices are all distinct points, the edges are simple curves, and no two edges meet one another except at their terminals. For example, K_4, the complete graph on four vertices, is planar, as Figure 4A shows. Many of the others, however, require proofs of unusual ingenuity and depth even in the two-dimensional case. Sometimes a plane solution may be readily extendible to higher dimensions, but sometimes just the opposite is true, and a three-dimensional or n-dimensional problem may be entirely different from its two-dimensional counterpart. Each new problem must be attacked individually. The continuing charm and challenge of the subject are at least in part due to the relative simplicity of the statements coupled with the elusive nature of their solutions. In 1893 the British mathematician J.J. Sylvester posed the question: If a finite set S of points in a plane has the property that each line determined by two points of S meets at least one other point of S, must all points of S be on one line? Sylvester never found a satisfactory solution to the problem, and the first (affirmative) solutions were published a half century later. Since then, Sylvester's problem has inspired many investigations and led to many other questions, both in the plane and in higher dimensions.

KEYWORDS;-PLANAR GRAPHS REDUCIBILITY, PREGEL RIVER, PACKING AND COVERING, PRISM AND TRUNCATED PYRAMID.

INTRODUCTION

Planar graphs
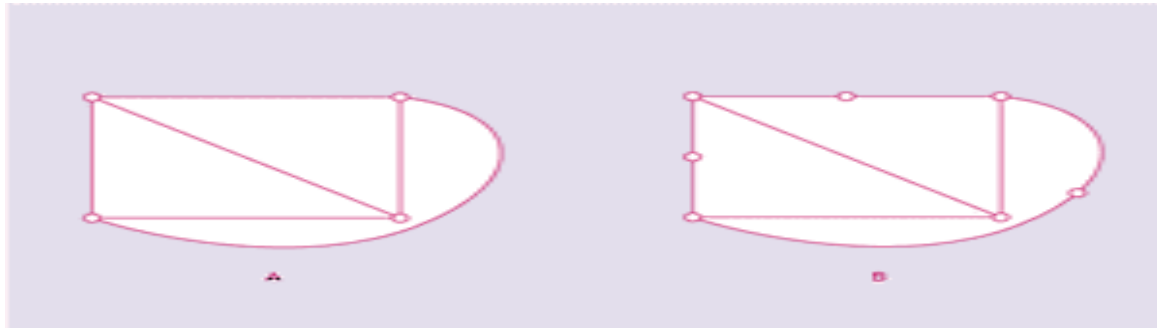
Two homeomorphic graphs A and B

A graph G is said to be planar if it can be represented on a plane in such a fashion that the vertices are all distinct points, the edges are simple curves, and no two edges meet one another except at their terminals. For example, K_4, the complete graph on four vertices, is planar, as Figure 4A shows.

Two graphs are said to be homeomorphic if both can be obtained from the same graph by subdivisions of edges. For example, the graphs in Figure 4A and Figure 4B are homeomorphic.

The $K_{m,n}$ graph is a graph for which the vertex set can be divided into two subsets, one with m vertices and the other with n vertices. Any two vertices of the same subset are nonadjacent, whereas any two vertices of different subsets are adjacent. The Polish mathematician Kazimierz Kuratowski in 1930 proved the following famous theorem:
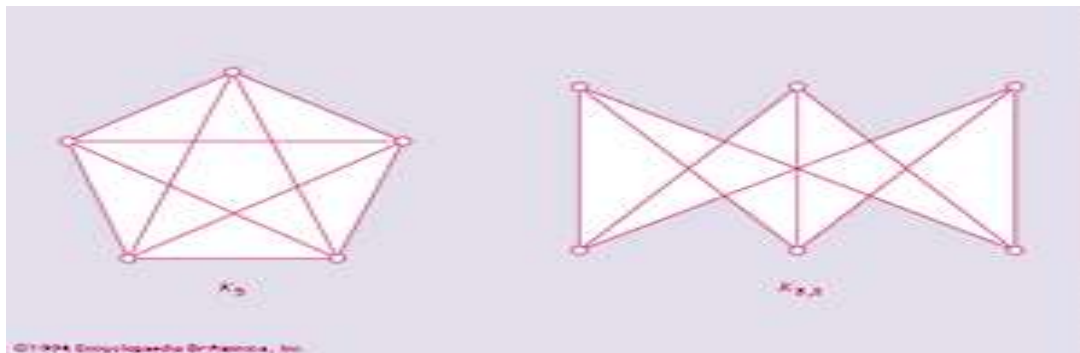
Two graphs important to planar properties

A necessary and sufficient condition for a graph G to be planar is that it does not contain a subgraph homeomorphic to either K_5 or $K_{3,3}$ shown in Figure 5.

An elementary contraction of a graph G is a transformation of G to a new graph G_1, such that two adjacent vertices u and υ of G are replaced by a new vertex w in G_1 and w is adjacent

in G_1 to all vertices to which either u or υ is adjacent in G. A graph G^* is said to be a contraction of G if G^* can be obtained from G by a sequence of elementary contractions. The following is another characterization of a planar graph due to the German mathematician K. Wagner in 1937. A graph is planar if and only if it is not contractible to K_5 or $K_{3,3}$.

The four-colour map problem

For more than a century the solution of the four-colour map problem eluded every analyst who attempted it. The problem may have attracted the attention of Möbius, but the first written reference to it seems to be a letter from one Francis Guthrie to his brother, a student of Augustus De Morgan, in 1852.

The problem concerns planar maps that is, subdivisions of the plane into nonoverlapping regions bounded by simple closed curves. In geographical maps it has been observed empirically, in as many special cases as have been tried, that, at most, four colours are needed in order to colour the regions so that two regions that share a common boundary are always coloured differently, and in certain cases that at least four colours are necessary. (Regions that meet only at a point, such as the states of Colorado and Arizona in the United States, are not considered to have a common boundary).

A formalization of this empirical observation constitutes what is called "the four-colour theorem." The problem is to prove or disprove the assertion that this is the case for every planar map. That three colours will not suffice is easily demonstrated, whereas the sufficiency of five colours was proved in 1890 by the British mathematician P.J. Heawood.

In 1879 A.B. Kempe, an Englishman, proposed a solution of the four-colour problem. Although Heawood showed that Kempe's argument was flawed, two of its concepts proved fruitful in later investigation.

One of these, called unavoidability, correctly states the impossibility of constructing a map in which every one of four configurations is absent (these configurations consist of a region with two neighbours, one with three, one with four, and one with five).

The second concept, that of reducibility, takes its name from Kempe's valid proof that if there is a map that requires at least five colours and that contains a region with four (or three or two) neighbours, then there must be a map requiring five colours for a smaller number of regions. Kempe's attempt to prove the reducibility of a map containing a region with five neighbours was erroneous, but it was rectified in a proof published in 1976 by Kenneth Appel

and Wolfgang Haken of the United States. Their proof attracted some criticism because it necessitated the evaluation of 1,936 distinct cases, each involving as many as 500,000 logical operations. Appel, Haken, and their collaborators devised programs that made it possible for a large digital computer to handle these details. The computer required more than 1,000 hours to perform the task, and the resulting formal proof is several hundred pages long.

Eulerian cycles and the Königsberg bridge problem

A **multigraph** G consists of a non-empty set $V(G)$ of vertices and a subset $E(G)$ of the set of unordered pairs of distinct elements of $V(G)$ with a frequency $f \geq 1$ attached to each pair. If the pair (x_1, x_2) with frequency f belongs to $E(G)$, then vertices x_1 and x_2 are joined by f edges.

An Eulerian cycle of a multigraph G is a closed chain in which each edge appears exactly once. Euler showed that a multigraph possesses an Eulerian cycle if and only if it is connected (apart from isolated points) and the number of vertices of odd degree is either zero or two.
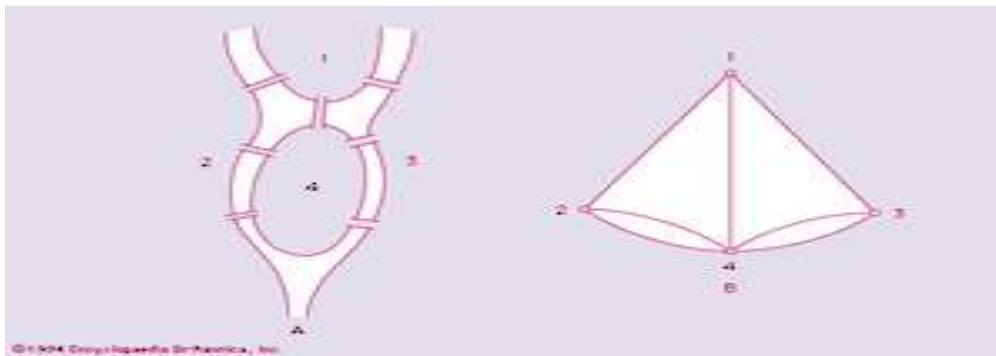
Seven bridges of Königsberg and multigraph.

This problem first arose in the following manner. The Pregel River, formed by the confluence of its two branches, runs through the town of Königsberg and flows on either side of the island of Kneiphof. There were seven bridges, as shown in Figure 6A. The townspeople wondered whether it was possible to go for a walk and cross each bridge once and once only. This is equivalent to finding an Eulerian cycle for the multigraph in Figure 6B. Euler showed it to be impossible because there are four vertices of odd order.

Directed graphs

A directed graph G consists of a non-empty set of elements $V(G)$, called vertices, and a subset $E(G)$ of ordered pairs of distinct elements of $V(G)$. Elements (x, y) of $E(G)$ may be called edges, the direction of the edge being from x to y. Both (x, y) and (y, x) may be edges.

A closed path in a directed graph is a sequence of vertices $x_0 x_1 x_2 \cdots x_n = x_0$, such that (x_i, x_{i+1}) is a directed edge for $i = 0, 1, \cdots, n-1$. To each edge (x, y) of a directed graph G there can be assigned a non-negative weight function $f(x, y)$. The problem then is to find a closed path in G traversing all vertices so that the sum of the weights of all edges in the path is a minimum.

This is a typical optimization problem. If the vertices are certain cities, the edges are routes joining cities, and the weights are the lengths of the routes, then this becomes the travelling-salesman problem—that is, can he visit each city without retracing his steps? This problem still remains unsolved except for certain special cases.

Combinatorial geometry

The name combinatorial geometry, first used by Swiss mathematician Hugo Hadwiger, is not quite accurately descriptive of the nature of the subject. Combinatorial geometry does touch on those aspects of geometry that deal with arrangements, combinations, and enumerations of geometric objects; but it takes in much more.

The field is so new that there has scarcely been time for it to acquire a well-defined position in the mathematical world. Rather it tends to overlap parts of topology (especially algebraic topology), **number theory, analysis,** and, of course, geometry. The subject concerns itself with relations among members of finite systems of geometric figures subject to various conditions and restrictions.

More specifically, it includes problems of covering, packing, symmetry, extrema (maxima and minima), continuity, tangency, equalities, and inequalities, many of these with special emphasis on their application to the theory of convex bodies. A few of the fundamental problems of combinatorial geometry originated with Newton and Euler. The majority of the significant advances in the field, however, have been made since the 1940s.

The unifying aspect of these disparate topics is the quality or style or spirit of the questions and the methods of attacking these questions. Among those branches of **mathematics** that interest serious working mathematicians, combinatorial geometry is one of the few branches that can be presented on an intuitive basis, without recourse by the investigator to any advanced theoretical considerations or abstractions.

Yet the problems are far from trivial, and many remain unsolved. They can be handled only with the aid of the most careful and often delicate reasoning that displays the variety and

vitality of geometric methods in a modern setting. A few of the answers are natural and are intuitively suggested by the questions.

Many of the others, however, require proofs of unusual ingenuity and depth even in the two-dimensional case. Sometimes a plane solution may be readily extendible to higher dimensions, but sometimes just the opposite is true, and a three-dimensional or *n*-dimensional problem may be entirely different from its two-dimensional counterpart. Each new problem must be attacked individually. The continuing charm and challenge of the subject are at least in part due to the relative simplicity of the statements coupled with the elusive nature of their solutions.

Some historically important topics of combinatorial geometry
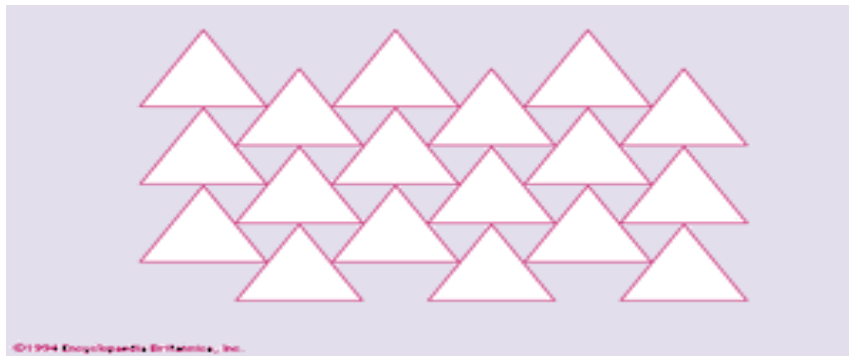
Packing and covering
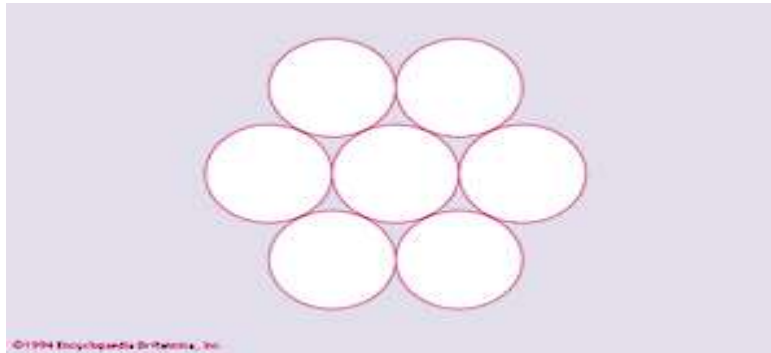
Packing of disks

It is easily seen that six equal circular disks may be placed around another disk of the same size so that the central one is touched by all the others but no two overlap (Figure 7) and that it is not possible to place seven disks in such a way. In the analogous three-dimensional situation, around a given ball (solid sphere) it is possible to place 12 balls of equal size, all touching the first one but not overlapping it or each other.

One such arrangement may be obtained by placing the 12 surrounding balls at the midpoints of edges of a suitable cube that encloses the central ball; each of the 12 balls then touches four other balls in addition to the central one. But if the 12 balls are centred at the 12 vertices of a suitable regular icosahedron surrounding the given ball, there is an appreciable amount of free space between each of the surrounding balls and its neighbours. (If the spheres have radius 1, the distances between the centres of the surrounding spheres are at least $2/\cos 18°$ = 2.1029 · · · .) It appears, therefore, that by judicious positioning it might be possible to have 13 equal non-overlapping spheres touch another of the same size. This dilemma between 12 and 13,

one of the first nontrivial problems of combinatorial geometry, was the object of discussion between Isaac Newton and David Gregory in 1694. Newton believed 12 to be the correct number, but this claim was not proved until 1953. The analogous problem in four-dimensional space was solved in 2003, the answer being 24.

The problem of the 13 balls is a typical example of the branch of combinatorial geometry that deals with packings and coverings. In packing problems the aim is to place figures of a given shape or size without overlap as economically as possibly, either inside another given figure or subject to some other restriction.


Covering of part of a plane with triangles

Problems of packing and covering have been the objects of much study, and some striking conclusions have been obtained. For each plane convex set *K*, for example, it is possible to arrange nonoverlapping translates of *K* so as to cover at least two-thirds of the plane; if *K* is a triangle (and only in that case), no arrangement of nonoverlapping translates covers more than two-thirds of the plane (Figure 8). Another famous problem was Kepler's conjecture, which concerns the densest packing of spheres. If the spheres are packed in cannonball fashion that is, in the way cannonballs are stacked to form a triangular pyramid, indefinitely extended then they fill π/Square root of√18, or about 0.74, of the space. In 1611 the German astronomer **Johannes Kepler** conjectured that this is the greatest density possible, but it was proved only in 1998 by the **American mathematician Thomas Hales.**

Covering problems deal in an analogous manner with economical ways of placing given figures so as to cover (that is, contain in their union) another given figure. One famous covering problem, posed by the French mathematician Henri Lebesgue in 1914, is still unsolved: What is the size and shape of the universal cover of least area? Here a convex set *C* is called universal cover if for each set *A* in the plane such that diam *A* 1 it is possible to move *C* to a suitable

position in which it covers A. The diameter diam A of a set A is defined as the least upper bound of the mutual distances of points of the set A. If A is a compact set, then diam A is simply the greatest distance between any two points of A. Thus, if A is an equilateral triangle of side 1, then diam $A = 1$; and if B is a cube of edge length 1, then diam $B =$ Square root of$\sqrt{3}$.

Polytopes
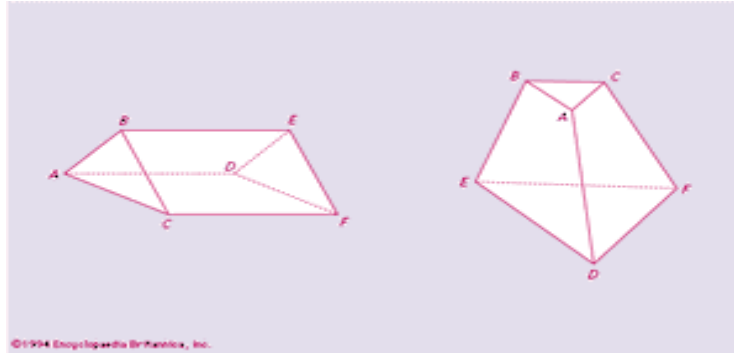
Prism and truncated pyramid

A (convex) polytope is the convex hull of some finite set of points. Each polytope of dimensions d has as faces finitely many polytopes of dimensions 0 (vertices), 1 (edge), 2 (2-faces), $\cdots$, d-1 (facets). Two-dimensional polytopes are usually called polygons, three-dimensional ones polyhedra.

Two polytopes are said to be isomorphic, or of the same combinatorial type, provided there exists a one-to-one correspondence between their faces, such that two faces of the first polytope meet if and only if the corresponding faces of the second meet. The prism and the truncated pyramid of Figure 9 are isomorphic, the correspondence being indicated by the letters at the vertices.

To classify the convex polygons by their combinatorial types, it is sufficient to determine the number of vertices υ; for each $\upsilon \geq 3$, all polygons with υ vertices (υ-gons) are of the same combinatorial type, while a υ-gon and a υ'-gon are not isomorphic if $\upsilon \neq \upsilon'$. Euler was the first to investigate in 1752 the analogous question concerning polyhedra.

He found that $\upsilon - e + f = 2$ for every convex polyhedron, where υ, e, and f are the numbers of vertices, edges, and faces of the polyhedron. Though this formula became one of the starting points of topology, Euler was not successful in his attempts to find a classification scheme for convex polytopes or to determine the number of different types for each υ. Despite efforts of many famous mathematicians since Euler (Steiner, Kirkman, Cayley, Hermes,

Brückner, to mention only a few from the 19th century), the problem is still open for polyhedra with more than 19 vertices.

The numbers of different types with four, five, six, seven, or eight vertices are 1, 2, 7, 34, and 257, respectively. It was established by American mathematician P.J. Federico in 1969 that there are 2,606 different combinatorial types of convex polyhedra with nine vertices. The number of different types for 18 vertices is more than 107 trillion.

The theory of convex polytopes has been successful in developments in other directions. The regular polytopes have been under investigation since 1880 in dimensions higher than three, together with extensions of Euler's relation to the higher dimensions. (The Swiss geometer Ludwig Schläfli made many of these discoveries some 30 years earlier, but his work was published only posthumously in 1901.) The interest in regular polyhedra and other special polyhedra goes back to ancient Greece, as indicated by the names Platonic solids and Archimedean solids.

Since 1950 there has been considerable interest, in part created by practical problems related to computer techniques such as linear programming, in questions of the following type: for polytopes of a given dimension d and having a given number υ of vertices, how large and how small can the number of facets be? Such problems have provided great impetus to the development of the theory. The U.S. mathematician Victor L. Klee solved the maximum problem in 1963 in most cases (that is, for all but a finite number of υ's for each d), but the remaining cases were disposed of only in 1970 by P. **McMullen, in the United States**, who used a completely new method.

Incidence problems

In 1893 the British mathematician J.J. Sylvester posed the question: If a finite set S of points in a plane has the property that each line determined by two points of S meets at least one other point of S, must all points of S be on one line? Sylvester never found a satisfactory solution to the problem, and the first (affirmative) solutions were published a half century later. Since then, Sylvester's problem has inspired many investigations and led to many other questions, both in the plane and in higher dimensions.

Helly's theorem

In 1912 Austrian mathematician Eduard Helly proved the following theorem, which has since found applications in many areas of geometry and analysis and has led to numerous

generalizations, extensions and analogues known as Helly-type theorems. If $K_1, K_2, \cdot\cdot\cdot, K_n$ are convex sets in d-dimensional Euclidean space E^d, in which $n \geq d + 1$, and if for every choice of $d + 1$ of the sets K_i there exists a point that belongs to all the chosen sets, then there exists a point that belongs to all the sets $K_1, K_2, \cdot\cdot\cdot, K_n$. The theorem stated in two dimensions is easier to visualize and yet is not shorn of its strength:

If every three of a set of n convex figures in the plane have a common point (not necessarily the same point for all trios), then all n figures have a point in common. If, for example, convex sets A, B, and C have the point p in common, and convex sets A, B, and D have the point q in common, and sets A, C, and D have the point r in common, and sets B, C, and D have the point s in common, then some point x is a member of A, B, C, and D.

Although the connection is often far from obvious, many consequences may be derived from Helly's theorem. Among them are the following, stated for $d = 2$ with some higher dimensional analogues indicated in square brackets:

A. Two finite subsets X and Y of the plane [d-space] may be strictly separated by a suitable straight line [hyperplane] if and only if, for every set Z consisting of at most 4 [$d + 2$] points taken from $X \cup Y$, the points of $X \cap Z$ may be strictly separated from those of $Y \cap Z$. (A line [hyperplane] L strictly separates X and Y if X is contained in one of the open half planes [half spaces] determined by L and if Y is contained in the other.)

B. Each compact convex set K in the plane [d-space] contains a point P with the following property: each chord of K that contains P is divided by P into a number of segments so the ratio of their lengths is at most $2d$.

C. If G is an open subset of the plane [d-space] with finite area [d-dimensional content], then there exists a point P, such that each open half plane [half space] that contains P contains also at least 1/3 [$1/(d + 1)$] of the area [d-content] of G.

D. If $I_1, \cdot\cdot\cdot, I_n$ are segments parallel to the y-axis in a plane with a coordinate system (x, y), and if for every choice of three of the segments there exists a straight line intersecting each of the three segments, then there exists a straight line that intersects all the segments $I_1, \cdot\cdot\cdot, I_n$.

Theorem **D** has generalizations in which kth degree polynomial curves $y = a_k x^k + \cdot\cdot\cdot + a_1 x + a_0$ take the place of the straight lines and $k + 2$ replaces 3 in the assumptions. These are important in the theory of best approximation of functions by polynomials.

Methods of combinatorial geometry

Many other branches of combinatorial geometry are as important and interesting as those mentioned above, but rather than list them here it is more instructive to provide a few typical examples of frequently used methods of reasoning. Because the emphasis is on illustrating the methods rather than on obtaining the most general results, the examples will deal with problems in two and three dimensions.

Exhausting the possibilities

Using the data available concerning the problem under investigation, it is often possible to obtain a list of all potential, a priori possible, solutions. The final step then consists in eliminating the possibilities that are not actual solutions or that duplicate previously found solutions. An example is the proof that there are only five regular convex polyhedra (the Platonic solids) and the determination of what these five are.

From the definition of regularity it is easy to deduce that all the faces of a Platonic solid must be congruent regular k-gons for a suitable k, and that all the vertices must belong to the same number j of k-gons. Because the sum of the face angles at a vertex of a convex polyhedron is less than 2π, and because each angle of the k-gon is $(k-2)\pi/k$, it follows that $j(k-2)\pi/k < 2\pi$, or $(j-2)(k-2) < 4$.

Therefore, the only possibilities for the pair (j, k) are (3, 3), (3, 4), (3, 5), (4, 3), and (5, 3). It may be verified that each of these pairs actually corresponds to a Platonic solid, namely, to the tetrahedron, the cube, the dodecahedron, the octahedron, and the icosahedron, respectively. Very similar arguments may be used in the determination of Archimedean solids and in other instances.

The most serious drawback of the method is that in many instances the number of potential (and perhaps actual) solutions is so large as to render the method unfeasible. Therefore, sometimes the exact determination of these numbers by the method just discussed is out of the question, certainly if attempted by hand and probably even with the aid of a computer.

Use of extremal properties
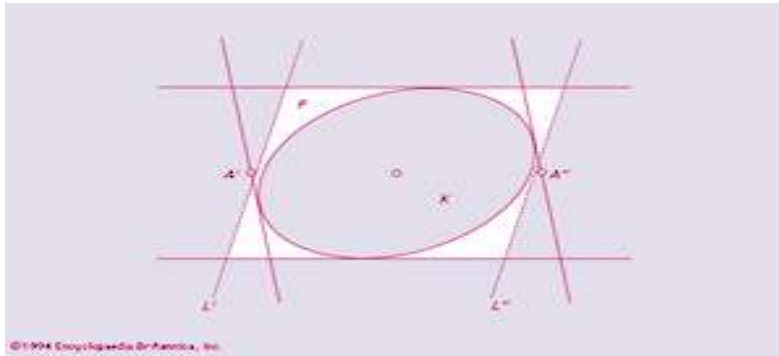
Example of theorem on extremal properties

In many cases the existence of a figure or an arrangement with certain desired properties may be established by considering a more general problem (or a completely different problem) and by showing that a solution of the general problem that is extremal in some sense provides also a solution to the original problem. Frequently there seems to be very little connection between the initial question and the extremal problem.

As an illustration the following theorem will be proved: If K is a two-dimensional compact convex set with a centre of symmetry, there exists a parallelogram P containing K, such that the midpoints of the sides of P belong to K. The proof proceeds as follows: Of all the parallelograms that contain K, the one with least possible area is labeled P_0.

The existence of such a P_0 is a consequence of the compactness of K and may be established by standard arguments. It is also easily seen that the centres of K and P_0 coincide. The interesting aspect of the situation is that P_0 may be taken as the P required for the theorem. In fact (Figure 10), if the midpoints A' and A of a pair of sides of P_0 do not belong to K, it is possible to strictly separate them from K by parallel lines L' and L that, together with the other pair of sides of P_0, determine a new parallelogram containing K but with area smaller than that of P_0. The above theorem and its proof generalize immediately to higher dimensions and lead to results that are important in **functional analysis.**

Sometimes this type of argument is used in reverse to establish the existence of certain objects by disproving the possibility of existence of some extremal figures. As an example the following solution of the problem of Sylvester discussed above can be mentioned. By a standard argument of projective geometry (duality), it is evident that Sylvester's problem is equivalent to the question: If through the point of intersection of any two of n coplanar lines, no two of which

are parallel, there passes a third, are the n lines necessarily concurrent? To show that they must be concurrent, contradiction can be derived from the assumption that they are not concurrent. If L is one of the lines, then not all the intersection points lie on L. Among the intersection points not on L, there must be one nearest to L, which can be called A. Through A pass at least three lines, which meet L in points B, C, D, so that C is between B and D. Through C passes a line L^* different from L and from the line through A. Since L^* enters the triangle ABD, it intersects either the segment AB or the segment AD, yielding an intersection point nearer to L than the supposedly nearest intersection point A, thus providing the contradiction.

The difficulties in applying this method are caused in part by the absence of any systematic procedure for devising an extremal problem that leads to the solution of the original question.

Use of transformations between different spaces and applications of Helly's theorem

The methods of proof in combinatorial geometry may be illustrated in one example—the proof of a theorem concerning parallel segments. Let the segment I_i have endpoints (x_i, y_i) and (x_i, y'_i), where $y_i\ y'_i$ and $i = 1, 2, \cdots, n$. The case that two of the segments are on one line is easily disposed of; so it may be assumed that $x_1, x_2, \cdots, x_n$ are all different. With each straight line $y = ax + b$ in the (x, y)-plane can be associated a point (a, b) in another plane, the (a, b)-plane. Now, for $i = 1, 2, \cdots, n$, the set consisting of all those points (a, b) for which the corresponding line $y = ax + b$ in the (x, y) plane meets the segment I_i can be denoted by K_i. This condition means that $y_i\ ax_i + b\ y'_i$ so that each set K_i is convex.

The existence of a line intersecting three of the segments I_i means that the corresponding sets K_i have a common point. Then Helly's theorem for the (a, b)-plane implies the existence of a point (a^*, b^*) common to all sets K_i. This in turn means that the line $y = a^*x + b^*$ meets all the segments $I_i, I_2, \cdots, I_n$, and the proof of theorem D is complete.

In addition to the methods illustrated above, many other techniques of proof are used in combinatorial   geometry,   ranging   from   simple mathematical induction to sophisticated decidability theorems of formal logic. The variety of methods available and the likelihood that there are many more not yet invented continue to stimulate research in this rapidly developing branch of **mathematics**

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