# A Study on Systematic Review of Fuzzy Variational Problems

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## Abstract

The study of fuzzy variational problems has received significant attention over the past decade due to its successful applications in numerous fields, such as image segmentation and optimal control theory. The fuzzy Euler-Lagrange equations provide the necessary optimality conditions for solving fuzzy variational problems explicitly and have been studied under several differentiability conditions. In this paper, we provide a systematic review to recap the history of variational principle in the calculus of variations and compare it with the existing techniques in the fuzzy setting. We begin with the preliminary concepts and definitions of fuzzy theory and scrutinize the Euler-Lagrange's strategy via systematically searched studies concerning fuzzy variational problems to highlight the importance of improving the existing methods. Finally, we set up the main open problems regarding the limitations of the current approaches, shedding light on future directions.

***Keywords:*** *Fuzzy; Euler; Variational.*

## 1. INTRODUCTION

Over the past decade, there has been significant development in the study of fuzzy variational problems due to their practicality in implementation. One of its crucial roles is in the field of image processing, which in turn is used in medical imaging, geographic imaging, and forensic science. The fuzzy variational problems are often solved numerically via Euler-Lagrange equations, Jacobi iterative techniques, or energy difference-based algorithms. One of such examples is the utilization of fuzzy Euler-Lagrange equations by Roul et al. to provide solution for cost minimization while optimizing the production rate. In the study, the considered problem is the minimization of a fuzzy cost functional, where the integrand is in terms of production and stock functions which are fuzzy mappings. The model allows the decision-makers to decide how to optimize the production rate given different stock levels while keeping the cost at its optimal value. On the other hand, multicriteria decision-making (MCDM) has been well studied under fuzzy set-theoretic models to analyze and rank alternatives from the finest to the poorest criteria concerning decision-maker preferences. Since most practical optimization problems can be modelled as fuzzy energy functionals, it is therefore advantageous to study fuzzy variational problems for decision-making. However, the existing studies in fuzzy variational problems are mostly aimed at obtaining the explicit solution by deriving the Euler-Lagrange equations in the fuzzy setting, which are essentially based on the intuition that every extremal problem will have a solution. This is in contrast to the standard variational principle in the calculus of variations that focuses on the existence of minimizers or more generally the existence of solutions.

Known as the classical method for solving variational problems in the calculus of variations, the history of the Euler-Lagrange approach can be traced back to one of the ancient problems in mathematics, namely, the isoperimetric problem or Dido's problem, that is to determine the curve of a given length which encloses a maximal area. Though a circle appeared to be an obvious solution to the isoperimetric problem, there was no rigorous mathematical proof until 1744 when Euler revisited the isoperimetric problem. The problem was eventually solved by means of the standard methods in calculus. However, Euler noted the complicated process of his geometrical approach, which may require a more straightforward strategy in acquiring the variational conditions. In 1755, Euler received a letter from a nineteen-year-old Lagrange who had found a more general approach to the problem, which is purely analytical. Dropping the geometrical approach, Euler and Lagrange developed a systematic way to solve such variational problems which includes Dido's problem, Galileo's brachistochrone problem that was formulated in 1638, Fermat's optical problem in 1662, and Newton's problem on floating bodies in 1685. This leads to what we know today as the Euler-Lagrange equation and hence the birth of the calculus of variations.

In view of modern calculus of variations, the Euler-Lagrange equation is usually regarded as the classical method, and one of its disadvantages is that it was established based on the intuition that every extremal problem will have a solution. In particular, while being a solution to the Euler-Lagrange equation is necessary for the minimization problem, it is far from being a sufficient condition. This approach seems to be inadequate since there is no guarantee that a minimizer will exist. Moreover, the solutions are assumed to be regular (usually or ). These limitations led to another class of methods, namely the direct method in the calculus of variations, which is a powerful abstract method for proving the existence of minimizers. This approach grew out of many people's work; in particular, its early development is often attributed to the works of Riemann, Hilbert, Weierstrass, and Tonelli. The essence of the direct method is to show the existence of a minimizer of a given integral functional and subsequently to prove its regularity. Interested readers on the history of calculus of variations may refer to the books of Giaquinta and Hildebrant, Goldstine, and Monna.

Instead of finding the minimizers of variational problems directly, the direct method focuses on finding these minimizers implicitly by considering minimizing sequences and the concept of lower semicontinuity of the given integral functional. In 1920, Tonelli proved that the functional is lower semicontinuous if and only if its integrand is twice continuously differentiable and convex in the last variable for the one-dimensional case. The work of Tonelli was further extended to the -dimensional case by McShane. However, though the regularity condition of the integrand assumed by McShane and Tonelli was sufficient, it was too strong to be true. Therefore, Serrin tried to relax the requirement of differentiability under the assumptions of continuity and nonnegativity. Later, instead of differentiability and continuity, Marcellini and Sbordone assumed the integrand to be a Carathéodory function (measurable in the first variable and jointly continuous in other variables) and gave the necessary and sufficient conditions for lower semicontinuity of the functional. This led to the general existence theorem with minimal regularity conditions. The variational principle in the calculus of variations has since been applied in studies that vary from the optimal design for thin films to the construction of an ideal column and from quantum field theory to softer spacecraft landings. Apart from that, the development in the calculus of variations has contributed to the advancement of multiple fields of mathematics, including topology, functional analysis, and partial differential equations.

One of the fundamental drawbacks of crisp functionals in the calculus of variations is the assumption that all the variables defining a particular energy functional would be accessible accurately. This is a significant restriction since, in most practical cases, not all factors are quantifiable. Recent research studies in image segmentation varying from the detection of the objects in an image to contouring vessel structures highlight similar drawbacks of the conventional crisp functional models, where they are highly dependent on the initial images and are unable to handle objects with ill-defined boundaries. Since not all results can be obtained in the crisp sense of Aristotelian logic due to the complexity of problems, the development of fuzzy functionals seems to be promising to handle such problems. In response to this, our study is aimed at discussing fuzzy functionals that consider both fuzzy logic and calculus of variations.

Many practical problems such as the propagation of wave, heat, fluid flow, and optimization theory are formulated using partial differential equations that can be solved by various means of numerical and analytical methods. One of such methods is the variational method using the Lagrangian density function to solve the corresponding partial differential equations. Though there have been studies focusing on solving the fuzzy partial differential equations to model uncertainties and to study the existence of their solutions, there has been little attention on the variational principle for solving fuzzy variational problems. In recent years, some studies have been dedicated to establishing the necessary optimality conditions of fuzzy variational problems for finding the extremals. However, the question remains to be considered is the existence of those extremals without resorting to the Euler-Lagrange equations.

## 2. STANDARD DEFINITIONS IN FUZZY THEORY

In this section, we present some standard definitions that follow from Dubo and Prade, Klir and Yuan, and Zimmermann. These definitions would be used throughout the paper.

*Definition 1 (crisp set).* A set A is called as a crisp set if there is no ambiguity regarding the inclusion or exclusion of its elements.

*Example 1.* Let A be the set of all natural numbers. Any real number will be either an element of the set or not.

*Definition 2 (fuzzy set and membership function).* Let be X the universe of discourse and represents its elements. Then, a fuzzy set in X , denoted by A , is defined using a set of ordered pairs as follows:

$$\widetilde{A} = \{\{x, \mu_{\widetilde{A}}(x)\} : x \in X\},$$

where $\mu_{\tilde{A}} : X \longrightarrow [0, 1]$ is called the membership function that gives a degree of membership to every element of ranging from to .

A fuzzy set is a mathematical way of representing ambiguous or subjective data obtained through human language. In a fuzzy set, an element can be partially contained in the set whenever its inclusion is ambiguous, and we represent it by the degree of membership that is between 0 and 1. For instance, consider a collection of tall 12 students of grade from a school. Now, the term "tall" seems subjective. A person with a height of 5.4 feet might be tall to someone who is 5 feet, but a 6.3 feet tall person may not consider feet to be tall. The word "tall" is ambiguous and subjective, and such a collection cannot be defined mathematically using the crisp set. Instead, it can be defined via fuzzy sets using linguistic language.

Similarly, let us consider the case of an automatic air-conditioning system along with a temperature sensor working over a fuzzy algorithm to monitor the speed of a motor that maintains an ideal temperature. A fuzzy inference system generally consists of fuzzification for giving fuzzy inputs, knowledge-based if-then rule evaluation followed by fuzzy outputs, and its defuzzification to obtain a single output of the aggregated fuzzy set. Fuzzification refers to the process of storing linguistic variables as fuzzy sets using membership functions. Figure 1 illustrates the linguistic representation of the air temperature using five fuzzy sets for temperature, namely, *Very cool*, *Cool*, *Ideal*, *Warm*, and *Very warm*.

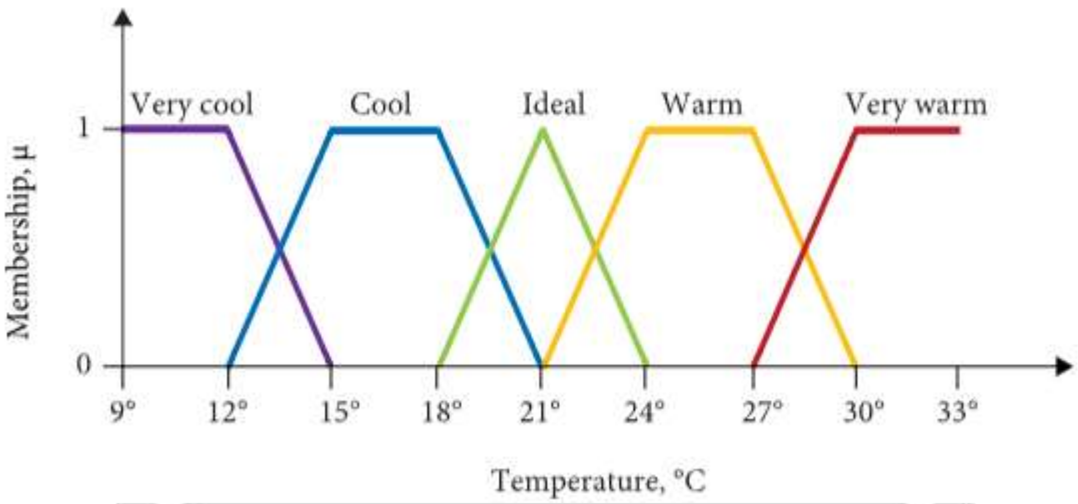


**Figure 1 Fuzzy sets with different membership functions corresponding to the range of temperature.**

## 3. EULER-LAGRANGE EQUATION FOR VARIATIONAL PROBLEMS

In this section, we first recall the classical approach of variational principle by means of the Euler-Lagrange equation. We scrutinize the strategy and working process of the Euler-Lagrange theorem and compare it with the existing studies concerning fuzzy variational principle to reveal the necessity for its improvement in addressing the current limitations. To begin, we consider the variational problem in the following form:

$$\inf_{u \in X} \left\{ I(u) = \int_{\Omega} f(x, u(x), \nabla u(x))\, dx \right\},$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded set, $x = (x_1, x_2, \cdots, x_n) \in \Omega$, $u : \Omega \longrightarrow \mathbb{R}^N$ is a continuously differentiable function, $X = \{u \in C^1(\Omega) : u = u_0 \text{ on } \partial\Omega\}$, and $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$ is a twice continuously differentiable function. In particular, we seek to minimize the above integral *I(u)* among all functions $u$ from the admissible space X . In the one-dimensional case and under ideal conditions, it is known that we can find the critical points of a function by letting its derivative to be equal to zero. Analogously, this can be done for solutions of smooth variational problems with vanishing functional derivative. To observe it, consider the above in one dimensional case, that is when n=N=1:

$$\inf_{u\in X}\left\{ I(u)=\int_a^b f\left(x,u(x),u'(x)\right)\,dx\right\}.$$

Conclusions and Future Directions

In the study of fuzzy variational principle, we observed that the necessary optimality conditions had been obtained by following the fundamental pathway established by Euler and Lagrange for problems in the calculus of variations. Considering twice differentiability, the variational principle via the Euler-Lagrange equations provides the necessary optimality conditions under various differentiability notions such as Buckley-Fuering and generalized Hukuhara (gH) derivatives. This was further extended and studied for fuzzy fractional variational principle under Caputo-type and gH Atangana-Baleanu derivatives. Most real-world problems such as image segmentation in medical and geographic imaging and optimal control theory generally focus on solving the corresponding energy integral, which corresponds to a fuzzy variational problem. This reflects the importance of studying and developing the field of fuzzy variational principle for better decision-making. Although decision-making is a well-established branch under fuzzy set-theoretic modeling, there is room for development in the fuzzy functional setting which can be applied to complex fuzzy mathematical problems when dealing with functionals. Some of the crucial questions are as follows:(1)What would the sufficient and necessary conditions be to ensure the existence of minimizers of the fuzzy variational problems?(2)How would the analogy of the direct method work for fuzzy variational problems?(3)Are fuzzy Euler-Lagrange equations and applicable to higher dimensions, or do they instead lead to complicated calculations?

Given the questions above, it is crucial to work on the sufficient conditions to ensure the existence and uniqueness of the minimizer. The classical approach of the fuzzy variational problem to find solutions via the Euler-Lagrange equations under twice differentiability seems restrictive. We can consider more relaxed regularity conditions to enlarge the space of admissible functions.

This systematic study also shows that the current approach for seeking a minimizer among functions satisfying the fuzzy Euler-Lagrange equations may lead to partial conclusions and complicated calculations in higher dimensions since the existence of a minimizer is assumed and not established beforehand. This limitation motivates us to study the variational principle to ensure the existence of minimizers for variational problems in the fuzzy setting. A similar analogy of the Euler-Lagrange theorem can be established in the fuzzy setting to obtain sufficient conditions for the existence of minimizer.

In terms of applications to real problems, it is necessary to establish the theory for functionals in fuzzy calculus of variations to find an optimized curve for optimal decision-making. For this purpose, one may utilize the calculus of variations' approach to work with higher dimensions without resulting in complex calculations.

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