A review on mathematical physics and graph theory

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ABSTRACT
We present some recent interesting results from mathematical physics and graph theory that has not appeared in social network . to understanding the roles of Eigen values and eigenvectors in network analysis which suggest the new directions and applications in network analysis. Further we extend the some of properties that can also be applied to the discrete Laplacian of a graph, Hilbert Nodal Theorem, and Cheeger's constant.

Key words : Graph theory, Laplacian operator, Tuttes alogrithem, Cheeger's constant.

Introduction :

The purpose of this note is to describe in passing some beautiful basic concepts interlacing mathematical physics, combinatorics and knot theory. There are many sources and the time constraint prevented from adding the references.

The Laplacian first appeared in solutions to the wave equation, and soon found application in many other areas of physics, such as the flow of heat (and other forms of diffusion). Many of the methods and functions of mathematical physics were developed to deal specifically with solutions to this operator, undoubtedly the most important and best understood mathematical idea in physics.

A graph is a collection of nodes that are joined by links, and in a random graph, the links are random. There are different types of graphs. In a static graph, the links are generated instantaneously, while in an evolving graph the links are generated sequentially.

In a simple graph, a given pair of nodes may be connected by a single link only, but in a multigraph they may be connected by multiple links. We consider the following random graph model. Starting with no links and N disconnected nodes, links are sequentially added between randomly selected pairs of nodes.

The use of graph theory in condensed matter physics, pioneered by many chemical and physical graph theorists (Harary, 1968; Trinajstić, 1992), is today well established; it has become even more popular after the recent discovery of graphene.

In this paper we will cover some of the most important areas of applications of graph theory in matter of physics. These include condensed matter physics, statistical physics, quantum electrodynamics, electrical networks and vibration problems and so on .

The Laplacian Operator in Physics

The Laplacian first appeared in solutions to the wave equation, and soon found application in many other areas of physics, such as the flow of heat (and other forms of diffusion). Many of the methods and functions of mathematical physics were developed to deal specifically with solutions to this operator, undoubtedly the most important and best understood mathematical idea in physics. For example, Fourier analysis was first developed to deal with the heat equation. The Laplacian may be constructed from two simpler operators, each a generalization of the derivative operator:
The gradient $\nabla\phi$ a unit vector. And $\nabla\varphi$ a scalr.

We notice that we have completely separated the properties of the Laplacian to be properties of the space in which it acts. Some of the spaces considered in exploring the properties of the Laplacian can be very complex indeed (having curved "surfaces", possibly with "holes" and "bridges"). In an important sense we can explore the properties of these "surfaces" (the general term in n-dimensions is "manifold") by exploring the properties of the Laplacian on these "surfaces".

Cheeger's Constant

Cheeger's constant is an isoperimetric property of a "surface" or manifold. An example of an isoperimetric problem is "what shape contains the largest volume for the least surface area"? In Euclidean space, the answer is a sphere (a circle in 2 dimensions, a hyper-sphere in n dimensions). Cheeger's constant is derived from the following problem:
Let E be a surface with area A, which divides a manifold into two parts with volumes V1 and V2.

The discrete Laplacian

We will derive the discrete Laplacian from discrete versions of the gradient and divergence. It will turn out to have a very simple form for graphs.

Discrete gradient

Here we want to model first differences, but our geometry is very simple: all distances are 1 and -1 . we only have differences between nodes that are connected. We can model the gradient by constructing a *signed incidence matrix* (or matrix of dyads) with edges represented by rows, and nodes represented by columns. We assign a pair of 1 and -1 for each edge arbitrarily (this will make no difference for the discrete Laplacian).

The language of graphs and networks

The first thing that needs to be clarified is that the terms graphs and networks are used indistinctly in the literature. In this part we will reserve the term graph for the abstract mathematical concept, in general referred to small, artificial formations of nodes and edges. The term network is then reserved for the graphs representing real-world objects in which the nodes represent entities of the system and the edges represent the relationships among them. Therefore, it is clear that we will refer to the system of individuals and their interactions as a 'social network' and not as a 'social graph'. However, they should mean exactly the same. For the basic concepts of graph theory the reader is recommended to consult the introductory book by Harary (1967). We start by defining a graph formally. In a **directed graph** the relation *E* is non-symmetric. In many physical applications the edges of the graphs are required to support weights, i.e., real numbers indicating a specific property of the edge. In this case the following more general definition is convenient. A **weighted graph** with *V* is a finite set of nodes, If the weights are natural numbers then the resulting graph is a multigraph in which there could be multiple edges between pairs of vertices. That is, if the weight between nodes and is it means that there are links between the two nodes.

Graphs and electrical networks

The relation between electrical networks and graphs is very natural and is documented in many introductory texts on graph theory. The idea is that a simple **electrical network** can be represented as a graph in which we place a fixed electrical resistor at each edge of the graph. Therefore, they can also be called **resistor networks**. Let us suppose that we connect a battery across the nodes and . There are several parameters of an electrical graph G (V,E) u v network that can be considered in terms of graph-theoretic concepts but we concentrate here in one which has important connections with other parameters of relevance in physics, namely the effective resistance (Doyle, Snell, 1984). Let us calculate the effective resistance *u*,*v* between two nodes by using the Kirchhoff and Ohm laws. For the sake of simplicity we always consider here resistors of 1 Ohm. In the simple case of a tree the effective resistance is simply the sum of the resistances along the path connecting *u* and *v* .

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