# An Economic Interpretation for Chemical Products by Duality in Linear Programming

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**ABSTRACT**

*This paper develops duality relations in linear programming which give new economic interpretations for chemical products. This method proposed in this paper is a dual version of the chemical products (aluminum sulphate & sulphuric acid) in simplex method. Duality in LP is often introduced through the relation between LP problems modeling different aspect of planning problem. The dual objective function is equal to the primal objective function from its resources. The observation in this study (Ethiopian chemical company located in Adama (Ethiopia)) was carried out, the dual results showed that the worth of one day of machine hrs on reaction indicated Birr 2141.86 and the worth of one day of demand for aluminum sulphate, Birr 2797.16.*

**Keyword:** - *linear programming, duality, simplex method, Birr, aluminum sulphate, sulphuric acid*
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## 1. Introduction

Duality is one of the most fundamental concepts in connection with linear programming and provides the basis for better understanding of LP model and their results. Duality in LP is introduced using chemical products upon the relationship between the primal and the dual problem seen from an economical perspective. [1]

In Linear programming, duality implies that each linear programming problem can be analyzed in two different ways but would have equivalent solutions.

The basic idea behind the duality theory is that every primal linear programming problem has an associated linear programming problem called its dual such that a solution to the primal linear programming also provides a solution to its dual. [2]

In linear programming, there are well-known economic interpretations for optimal solutions to the dual problem. With those interpretations in mind, dual optimal solutions have been termed “shadow price vectors” [3] and “equilibrium price vectors” [4, 5].

Duality in linear programming find the marginal value (also known as shadow price) of each resources. This value reflects an additional cost to be paid to obtain one additional unit of the resource to get the optimal value of objective function under resource constraints.

The shadow price is defined as the rate of change in the optimal objective function value with to the unit change in the availability of a resource. [6]

$$\text{Shadow} \qquad \frac{\text{Change in optimal objective function value}}{\text{unit change in the availability of resource}}$$

## 2. General formulation of dual LP problem

The typical formulation of an LP problem with n nonnegative variables and m inequality constraints is

Maximize $Z_x = c_1x_1 + c_2x_2 + ... + c_nx_n$

Subject to the constraints

$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \le b_1$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \le b_2$$
.
.
.
$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \le b_m$$
$$x_1, x_2, ..., x_n \ge 0$$

Then the corresponding dual LP problem,

Minimize $Z_y = b_1y_1 + b_2y_2 + ... + b_my_m$

Subject to the constraints

$$a_{11}y_1 + a_{21}y_2 + ... + a_{m1}x_m \ge c_1$$
$$a_{12}y_1 + a_{22}y_2 + ... + a_{m2}x_m \ge c_2$$
.
.
.
$$a_{1n}y_1 + a_{2n}y_2 + ... + a_{mn}x_m \ge c_{1n}$$
$$y_1, y_2, ..., y_m \ge 0$$

From these expressions of parameters of both primal and dual problems, it is clear that for the unit of measurement to be consistent, the dual variable ($y_i$)must be expressed in terms of return (or worth) per unit of resource *i*. This is called dual price (simplex multiplier, or shadow price) of resource *i*.

## 3. Economic interpretation of dual

Consider the following problem:
Maximum

$$c_1x_1 + ..... + c_nx_n$$

Subject to all $x_i \ge 0$

$$a_{11}x_1 + ... + a_{1n}x_n \le b_1$$
....
$$a_{m1}x_1 + ... + a_{mn}x_n \le b_m$$

Economic interpretation
n = economic activities, m resources
$c_j$ = revenue per unit of activity *j*
$b_i$ = maximum availability of resource *i*
$a_{ij}$ = consumption of resource *i* per unit of activity *j*

*The dual*
Minimize

$$b_1y_1 + ... + b_my_m$$

Subject to all $y_i \ge 0$

$$a_{11}y_1 + ... + a_{1m}y_m \ge c_1$$
....
$$a_{n1}y_1 + ... + a_{nm}y_m \ge c_n$$

### 3.1 Interpreting the dual variables

If ( $x_1,..., x_n$ ) is optimal for the primal, and ( $y_1,..., y_n$ ) is optimal for the dual, then we know:

$$c_1x_1 + ... + c_nx_n = b_1y_1 + ... + b_my_m$$

Left-hand side: Maximal revenue
Right-hand side:
$\sum_{\text{resource i}}$ (availability of resource *i*) × (revenue per unit of resource *i*)
In other words: Value of $y_i$ at optimal is dual price of resource *i*
Anyway from optimality, we have

$$c_1x_1 + ... + c_nx_n < b_1y_1 + ... + b_my_m$$

Left-hand side: current (suboptimal) revenue
Right-hand side: $\sum_{\text{resource i}}$ (worth of resource *i*)
Solution is not optimal because resources are not being fully utilized

*3.2 Interpreting the dual constraints*

If ( $x_1,..., x_n$ ) is feasible (not necessarily optimal) for the primal, and ( $y_1,..., y_n$ ) is the corresponding collection of dual values, then we know:
Currently objective coefficient of xj
= (Left-hand side of dual constraint j) – (Right-hand side)
= ( $a_1y_1 + ... + a_{mj}y_j$ )- $c_j$

$c_j$ is a measure of revenue per unit (of activity *j*)

So $(a_{1j}y_1 + ... + a_{mj}y_j)$ is an imputed (implicit) cost per unit (of act. *j*)

Also, $y_i$ is imputed cost per unit of resource I in a unit of activity *j*

If objective coefficient of $x_j$ (= cost – revenue, = reduced cost) is strictly negative, then revenue > cost, so it makes since to increase activity j – this is the pivoting process of the simplex method.

## 4. DATA PRESENTATION AND ANALYSIS

A chemical industry owned by the Ethiopian Government manufactures two major products, aluminum sulphate and sulphuric acid each of which passes through three different processes: reaction, filtration, and evaporation.

Table 1: Number of hours of machine time on three processes and the demand of items

| Products | Machine hours per ton per day | | | Produced/day (in ton) | Profit/per ton (in Birr) |
|---|---|---|---|---|---|
| | Reaction | Filtration | Evaporation | | |
| Aluminum Sulphate | 0.45 | 0.2 | 0.1 | 20 | 3760.66 |
| Sulphuric acid | 2.15 | 2.15 | 2.15 | 51.5 | 4604.58 |
| Available resource | 24 | 24 | 24 | | |

LP model

Maximize $Z_{\max} = 3760.66x_1 + 4604.58x_2$

Subject to:

$$0.45x_1 + 2.15x_2 \leq 24$$ hrs. (machine hrs. on reaction)

$$0.2x_1 + 2.15x_2 \leq 24$$ hrs. (machine hrs. on filtration)

$$0.1x_1 + 2.15x_2 \leq 24$$ hrs. (machine hrs. on evaporation)

$$x_1 \leq 20$$ tons (demand for aluminum sulphate per day)

$$x_2 \leq 51.5$$ (sulphuric acid produced per day) , $x_1, x_2 \geq 0$

By applying the MS-Excel Solver, the results are $x_1$ =20, $x_2$ = 6.98, Max Z = 107,347.907. Therefore, Company produces 20 tons of aluminum sulphate per day and 6.98 tons of sulphuric acid per day in order to get a maximum daily profit of (Birr 107,347.907) kyats 5,577,562.42 (1Birr =51.96 kyats).

By using duality,

LP model

Minimum $Z_{\min} = 24y_1 + 24y_2 + 24y_3 + 20y_4 + 51.5y_5$

Subject to:

$$0.45y_1 + 0.2y_2 + 0.1y_3 + y_4 \geq 3760.66$$

$$2.15y_1 + 2.15y_2 + 2.15y_3 + y_5 \geq 4604.58$$

And $y_1, y_2, y_3, y_4, y_5 \geq 0$

By applying the MS-Excel Solver, the results are $y_1$ = 2141.86, $y_4$ = 2797.16, Min Z = 107,347.907. Therefore, the optimal value of dual variable $y_1$ = 2141.86 represents the per unit price (worth or marginal price) of the first resource (i.e. machine hrs on reaction). This means that marginal contribution of machine on reaction to the total profit is Birr 2141.86. In other words, if one productive unit is removes from the right hand side of the first constraint, the total contribution would reduce by Birr 2141.86.

The optimal value of dual variable $y_4$ = 2797.16 represents the per unit price (worth or marginal price) of the fourth resource (i.e. demand for aluminum sulphate per day). This means that marginal contribution of demand for aluminum sulphate per day to the total profit is Birr 2797.16. In other words, if one productive unit is removes from the right hand side of the fourth constraint, the total contribution would reduce by Birr 2797.16.

## 5.Conclusion

Applying duality theory of linear programming is an interesting concept in operation research. It is advantageous to solve the dual of a primal that has a less number of constraints because the number of constraints usually equals the number of iterations required to solve the problem. The dual variables provide an important economic interpretation of the final solution of an LP problem. In the area like economic, it is highly helpful in obtaining future decision in the activities being programmed. Economic interpretation can be made and shadow prices can be determined enabling managers to take further decisions.

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