Analytic solving of the « Cauchy problem for heat equation with fractional Laplacian »using the generalized function theory

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Abstract

We try to give one method of resolution of the Cauchy problem for heat equation with fractional Laplacian by exploiting the generalized function theory

Keys words

Fractional Laplacian ,generalized function,Cauchy Problem

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1. Introduction

This paper concerns the Cauchy problem for the following heat equation

$$\begin{cases} f(t,x)+(-\Delta)^{\frac{\beta}{2}} f(t,x)=S(t,x) \\ f(0,x)=f_0 \end{cases} \quad t>0\,, x\in\mathbb{R}^n,\ 0<\beta<1 \ (1.1)$$

And the aim is to show, how the theory of distribution solves this equation, obviously we must give the solution.For that we will have to recall the mathematical theory necessary for this problem,the generalized functions theory and the semigroup,see [10]. All the same, there are certain mathematical properties that must be remembered during the solving

This paper is organized as follows. Section 2 presents some important definitions and proprieties used in this paper.The section section 3 will be divided into 3 parts, the first will be devoted to finding the elementairy solution of the heat equation with fractional operator and the resolution of (1) when S is genarlized function with compact support. The second part will be for the equation (1.1) without second member.The last part is for (1.1) when $S\in C(\mathbb{R}_+, L^2(\mathbb{R}^N)\cap L^1(\mathbb{R}^N))$

2. Preliminairy

We consider the fractional Laplace operator $L=-(-\Delta)^{\frac{\beta}{2}}$ in $\mathbb{R}^N$, with $\beta\in\left]0,1\right[$ and $N\in\mathbb{N}^*$.

Definition 1

For the definition of L *that: we will use a Fourier definition :*

Let $\mathcal{X}$ *be any of the spaces* L^p *,* $p\in[1,2]$ *and let* $f\in\mathcal{X}$ *hence* $Lf\in\mathcal{X}$ see[12]

$$\mathcal{F}(Lf)(\xi)=-|\xi|^{\beta}\mathcal{F}(f)(\xi) \qquad (2.1)$$

To define the fractional Laplacian for less regular functions, we need to understand what $(-\Delta)^{\frac{\beta}{2}}$ is in the sense of distributions. One may then think in the following way. For $u\in\mathcal{S}'$

(a tempered distribution) and $f \in \mathcal{S}$ one could define the distribution $(-\Delta)^{\frac{\beta}{2}}u$ as $\left\langle (-\Delta)^{\frac{\beta}{2}}u, f \right\rangle = \left\langle u, (-\Delta)^{\frac{\beta}{2}} f \right\rangle$.The problem here is that $(-\Delta)^{\frac{\beta}{2}} f \notin \mathcal{S}$, so this identity makes no sense for $u \in \mathcal{S}'$. First we need to characterize the set $(-\Delta)^{\frac{\beta}{2}}(\mathcal{S})$. See [13],[14],[15]

Proprety 1 *Let. $f \in \mathcal{S}$,then $(-\Delta)^{\frac{\beta}{2}} f \in \mathcal{S}_{\frac{\beta}{2}}$ belongs to the class $\mathcal{S}_{\frac{\beta}{2}}$ defined by*

$\mathcal{S}_{\frac{\beta}{2}} = \{\psi \in C^{\infty}(\mathbb{R}^N): (1+|x|^{n+\beta})D^{\gamma}\psi(x) \in L^{\infty}(\mathbb{R}^N), for\ every\ \gamma \in \mathbb{N}^N\}$ see[10]

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The class of $\mathcal{S}_{\frac{\beta}{2}}$ of Lemma 1 is endowed with the topology induced by the countable family of seminorms

$$\rho_{\gamma}(\psi) = \sup_{x \in IR^N} \left|(1+|x|^{N+\beta})D^{\gamma}\psi(x)\right| , \ \gamma \in \mathbb{N}^N$$

.

Denote by $\mathcal{S}'_{\frac{\beta}{2}}$ the dual space of$\mathcal{S}_{\frac{\beta}{2}}$. Observe that $\mathcal{S} \subset \mathcal{S}_{\frac{\beta}{2}}$ so that $\mathcal{S}'_{\frac{\beta}{2}} \subset \mathcal{S}'$. The suitable space for the distributional definition of the fractional Laplacian is $\mathcal{S}'_{\frac{\beta}{2}}$

Definition 2.

Let 'u $\in \mathcal{S}'_{\frac{\beta}{2}}$. We define $(-\Delta)^{\frac{\beta}{2}}u \in \mathcal{S}'$ as $\left\langle (-\Delta)^{\frac{\beta}{2}}u, f \right\rangle = \left\langle u, (-\Delta)^{\frac{\beta}{2}} f \right\rangle$ (2.2)

for every $f \in \mathcal{S}$.

Proprety 2 see[8]

For $x \in \mathbb{R}^N$ and t>0,let $S_{\beta}(t)(x) = S(t,x) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{i\xi.x - t|\xi|^{\beta}} d\xi$. (2.3)

For any function u_0 defined on $\mathbb{R}^N$, if the convolution product is valid , $e^{-t(-\Delta)^{\frac{\beta}{2}}}$ is a strongly continuous semigroup on $L^p(\mathbb{R}^N)$, p > 1, generated by the fractional power . $-(-\Delta)^{\frac{\beta}{2}}$ *and*

$S(t,x) * u_0 = e^{-t(-\Delta)^{\frac{\beta}{2}}} u_0 = P(t)u_0$

Moreover S_{β} satisfies :

- $S_{\beta}(1) \in L^1(\mathbb{R}^1) \cap L^{\infty}$ (2.4)
- $S_{\beta}(t,x) \ge 0$ (2.5)
- $\int_{IR^N} S_{\beta}(t,x)dx = 1$ (2 .6) *for all $x \in \mathbb{R}^N$ and t>0 ,it means $S_{\beta}(t,x) \in L^1(IR^N)$ for all t>0*
- $\left\| e^{-(-\Delta)^{\frac{\beta}{2}}} v \right\|_q \le C t^{\frac{N}{\beta}(\frac{1}{r}-\frac{1}{q})} \|v\|_r$ (2.7) *for all $v \in L^r(\mathbb{R}^N)$ and all $1 \le r \le q \le \infty$, t>0*

- using Young's inequality for the convolution :

$\left\| e^{-t(-\Delta)^{\frac{\beta}{2}}} v \right\|_q \|S_{\beta}(x,t) * v\|_q \le \|S_{\beta}(t)\|_1 \|v\|_q$ *for all $v \in L^q(\mathbb{R}^N)$and all$1 \le q \le \infty$,t>0*(2.8)

3 The resolution

3.1 Elementary solution of heat fractional opérator

Proposition 3 (Elementary solution of heat fractional opérator)

For all integer $N \geq 1$*, the equation* $(\partial_t + (-\Delta)_x^{\frac{\beta}{2}})u = \delta_{(t,x)=(0,0)}$ (3.1) *has an unique solution* E_N *in* $\mathcal{S}'(\mathbb{R}_t \times \mathbb{R}_x^N)$ *, the set of the tempered distribution. Its support is* $\operatorname{supp}(E_N) \subset \mathbb{R}_t^+ \times \mathbb{R}^N$
and its partial Fourier transform into the variable x is the function

$$\hat{E}_N(t,\xi) = 1_{\mathbb{R}_+^*}(t)\, e^{-t|\xi|^\beta} \quad , \xi \in \mathbb{R}^N \quad (3.2)$$

Proof :

Let us find, for all $N \geq 1$, a distribution $E_N \in \mathcal{S}'(\mathbb{R}_t \times \mathbb{R}^N)$ such that:

- E_N is function defined on $\mathbb{R}_t \times \mathbb{R}_x^N$ and $\operatorname{supp}(E_N) \subset \mathbb{R}_t^+ \times \mathbb{R}^N$
- $E_N(t,x) \in L^1(\mathbb{R}^N)$ for all t>0 such as $(\partial_t + (-\Delta)_x^{\frac{\beta}{2}})E_N = \delta_{(t,x)=(0,0)}$ in $\mathcal{S}'(\mathbb{R}_t \times \mathbb{R}^N)$

It is obvious that if E is an elementary solution of the heat fractional operator then E + Const. is also an elementary solution.

We are going to remove this indetermination thanks to the following support condition

$$supp(E_N) \subset \mathbb{R}_+ \times \mathbb{R}^N$$

Apply to the tempered distribution E_N the partial Fourier transformation into the variable x, which we will denote by $\hat{E}_N$, we will also note ξ the dual Fourier variable of x.

Note that $E_N(t,x) \in L^1(\mathbb{R}^N)$ for all t> 0 hence we can use (2.1)Then we have

$$\partial_t \hat{E}_N + |\xi|^\beta \hat{E}_N = \delta_{t=0} \otimes 1 \text{ in } \mathcal{S}'(\mathbb{R}_t \times \mathbb{R}^N) \quad (3.3)$$

$$supp\,(\hat{E}_N) \subset \mathbb{R}_+ \times \mathbb{R}^N$$

In particular, the restriction of $\hat{E}_N$ to $\mathbb{R}_+^* \times \mathbb{R}^N$ checks

$$\partial_t \hat{E}_N |_{\mathbb{R}_+^* \times \mathbb{R}^N} + |\xi|^\beta \hat{E}_N |_{\mathbb{R}_+^* \times \mathbb{R}^N} = 0 \text{ in } \mathcal{S}'(\mathbb{R}_+^* \times \mathbb{R}^N) \quad (3.4)$$

This suggests choosing the distribution $\hat{E}_N |_{\mathbb{R}_+^* \times \mathbb{R}^N}$ as being defined by a function of form

$$(t,\xi) \to C(\xi) e^{-t|\xi|^\beta}$$

And since the $\hat{E}_N$ distribution is supported in $\mathbb{R}_+ \times \mathbb{R}^N$, it is natural to find the distribution $\hat{E}_N$ as being globally defined by the function

$$\hat{E}_N(t,\xi) = 1_{\mathbb{R}_+^*}(t)\, C(\xi) e^{-t|\xi|^\beta} \quad (t,\xi) \in \mathbb{R} \times \mathbb{R}^N \quad (3.5)$$

If we take $\hat{E}_N(t,\xi) = 1_{\mathbb{R}_+^*}(t)\, e^{-t|\xi|^\beta}$ .so $E_N(t,x) = S(t,x) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{IR^N} e^{i\xi.x - t|\xi|^\beta} d\xi$ for t>0 and

$\operatorname{Supp}(E_N) \subset \mathbb{R}_+^* \times \mathbb{R}^N$ .Hence $E_N(t,x) \in L^1(\mathbb{R}^N)$ for all t>0.(see (2.6))

Let us pass to the proof of uniqueness. Suppose there is another tempered distribution E_N^1 satisfying the same properties as E_N, and denote by $F_N = E_N - E_N^1$ We can easily verify that F_N satisfies the hypotheses of the lemma below, from which we deduce that $F_N = E_N - E_N^1 = 0$, that is to say that $E_N = E_N^1$

Lemma 4

Let $F \in \mathcal{S}'(\mathbb{R}_t \times \mathbb{R}^N)$ such that $(\partial_t + (-\Delta)^{\frac{\beta}{2}})F = 0$ in $\mathcal{S}'(\mathbb{R}_t \times \mathbb{R}^N)$ and $supp(F_N) \subset \mathbb{R}^+ \times \mathbb{R}^N$ then $F = 0$ see [1]

3.2 Case where the initial data are are compact support distributions

Lemma 5 (Solution in $\mathcal{D}'$ of an ODE, Cauchy problem)

Let $a \in \mathbb{C}, S \in C_c(\mathbb{R}_+;\mathbb{C})$ and $u_0 \in \mathbb{C}$,let $u \in C^1(\mathbb{R}_+;\mathbb{C})$ the only solution of the Cauchy problem:

$$u' + au = S \text{ for t>0}$$
$$u(0) = u_0$$

Let U be a function defined on $\mathbb{R}^*$ with values in $\mathbb{C}$ defined by:

$$U(t) = u(t) \text{ for t>0 and } U(t) = 0 \text{ for t<0}$$

Then

(a)The function locally bounded $1_{\mathbb{R}_+}(t)E_a(t) = 1_{\mathbb{R}_+}(t)e^{-at}$ is an elementary solution of the differential operator $\frac{d}{dt} + a$ on $\mathbb{R}$

(b) The function locally bounded U verifies

$$U = \left(1_{\mathbb{R}_+}E_a\right) * (u_0\delta_0 + 1_{\mathbb{R}_+}S) \text{ dans } \mathcal{D}'(\mathbb{R})$$

(c) The function locally bounded U defines the unique distribution on $\mathbb{R}$ verifying

$$U' + aU = 1_{\mathbb{R}_+}S + u_0\delta_0$$
$$supp(U) \subset \mathbb{R}_+$$

Proof: see [1]

Proposition 4(Solution in $\mathcal{S}'$ of Cauchy problem for heat fractional equation when the initial data are in $\mathcal{E}'$)

Let be $N \geq 1$,an initial data $f_0 \in \mathcal{E}'(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and a source term $S \in \mathcal{E}'(\mathbb{R}_+^* \times \mathbb{R}^N)$ such $S(t,.) \in L^1(\mathbb{R}^N)$ for all t>0, both with compact support

There is then a unique solution within the meaning of the tempered distributions of the Cauchy problem for the heat fractional equation with initial data f_0 end and second member S.

This solution f is given by the formula

$$f = E_N * (\delta_{t=0} \otimes f_0 + S)$$

from which we deduce in particular, when $S \in C_c^\infty(\mathbb{R}_+^* \times \mathbb{R}^N)$, than

$$f|_{\mathbb{R}_+^* \times \mathbb{R}^N} \in C^\infty(\mathbb{R}_+^* \times \mathbb{R}^N)$$

Lemma 6

Let be $N \geq 1$,an initial data $f_0 \in \mathcal{E}'(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and a source term $S \in \mathcal{E}'(\mathbb{R}_+^* \times \mathbb{R}^N)$ such $S(t,.) \in L^1(\mathbb{R}^N)$ for all t>0, both with compact support.

We say that a distribution $F \in \mathcal{S}'(\mathbb{R}_t \times \mathbb{R}_x^N)$ such that $F(t,.) \in L^1(\mathbb{R}^N)$ is a solution within the meaning of (tempered) distributions of the Cauchy problem

$$\partial_t f(x,t) + (-\Delta_x)^{\frac{\beta}{2}} f(t,x) = S(t,x) \text{ for } t>0,\ x \in \mathbb{R}^N,\ f(t,) \in L^1(\mathbb{R}^N) \text{ for all } t>0$$
$$f(x,0) = f_0$$

when the distribution F verifies

$$\partial_t F + (-\Delta_x)^{\frac{\beta}{2}} F = \dot{S} + \delta_{t=0} \otimes f_0 \text{ in }'(\mathbb{R} \times \mathbb{R}^N),\ supp(F) \subset \mathbb{R}_+ \times \mathbb{R}^N$$

where $\dot{S}$ is the prolongation of S by 0 for $t \leq 0$

$$\left\langle \dot{S}, \phi \right\rangle_{\mathcal{E}'(\mathbb{R}\times\mathbb{R}^N), C^\infty(\mathbb{R}\times\mathbb{R}^N)} = \left\langle S, \phi_{|\mathbb{R}_+^*\times\mathbb{R}^N} \right\rangle_{\mathcal{E}'(\mathbb{R}_+^*\times\mathbb{R}^N), C^\infty(\mathbb{R}_+^*\times\mathbb{R}^N)}$$

Proof the lemma 6

For $\phi \in \mathcal{S}'(\mathbb{R}_t \times \mathbb{R}^N)$, denote by $\hat{\phi}$ the partial Fourier transform at x of the distribution ϕ, and ξ the dual Fourier variable of x. By applying the partial Fourier transform $\phi \mapsto \hat{\phi}$ to each member of the two equalities occurring in the Cauchy problem, we end up

$$\partial_t \hat{f} + |\xi|^\beta \hat{f} = \dot{S} \text{ for all t>0, } \xi \in \mathbb{R}^N$$

$$\hat{f}_{|t=0} = \hat{f}_0$$

When $\hat{f}$ is a function of class C^1 in (t,ξ) then the Cauchy problem above corresponds to a family of indexed ordinary differential equations by $\xi \in \mathbb{R}^N$.
The case of ordinary differential equations in the lemma 5 suggests that the above Cauchy problem after Fourier transformation partial in x admits the following formulation in the sense of the distributions:

$$\partial_t \hat{F} + |\xi|^\beta \hat{F} = \dot{S} + \delta_{t=0} \otimes \hat{f}_0 \quad \text{in } \mathcal{S}'(\mathbb{R} \times \mathbb{R}^N)$$

$$supp(\hat{F}) \subset \mathbb{R}_+ \times \mathbb{R}^N$$

By returning to physical variables by inverse Fourier transformation partial in the variable x, end up with the following definition of the notion of solution in the sense of distributions for a Cauchy problem with Fractional Laplacian Q.E .D

Proof of the proposition 4 :
According to the lemma 6. Say that f is the solution of the above Cauchy problem to direction of the tempered distributions, that is to say that $f \in \mathcal{S}'(\mathbb{R}_t \times \mathbb{R}^N)$ and $f(t,.) \in L^1(\mathbb{R}^N)$ and check

$$\partial_t f + (-\Delta)^{\frac{\beta}{2}} f = \delta_{t=0} \otimes f_0 + \dot{S}, \; x \in \mathbb{R}^N, t>0$$

$$supp(f_0) \subset \mathbb{R}_+ \times \mathbb{R}^N$$

where $\dot{S}$ is the extension of the distribution with compact support S by 0 in $\mathbb{R}_- \times \mathbb{R}^N$
,
Check that the proposed formula does provide a solution to the problem of Cauchy considered:

- Since f_0 and S are compactly supported distributions, the distribution $\delta_{t=0} \otimes f_0 + \dot{S}$ also has compact support.
- According to [G] the distribution $E_N * (\delta_{t=0} \otimes f_0)$coincides with the function $E_N(t,.) * f_0(.)$ when $f_0 \in C_c(\mathbb{R}^N)$ and t>0.This property still remains true if $f_0 \in L^1(\mathbb{R}^N)$,because:

We know that $C_c(\mathbb{R}^N)$ is dense in $L^1(\mathbb{R}^N)$, then for all $f_0 \in L^1(\mathbb{R}^N)$, there exists a sequence $(f_0^n)$of $C_c(\mathbb{R}^N)$ which converges to $f_0 \in L^1(\mathbb{R}^N)$. According to (2.8):
$\|E_N(t,.) * (f_0^n - f_0)\|_1 \le \|f_0^n - f_0\|_1$when t>0, then the sequence $(E_N(t,.) * f_0^n(.))$ converges to $E_N(t,.) * f_0(.)$ in $L^1(\mathbb{R}^N)$. Thus the convergence is also valid in $\mathcal{D}'(\mathbb{R}^N)$ when t>0. Hence $(E_N * (\delta_{t=0} \otimes f_0^n))$ converges to $E_N(t,.) * f_0(.)$ when t>0.
But we have, when $\phi \in C_c(\mathbb{R}_t \times \mathbb{R}_x^N)$:
$|\langle \delta_{t=0} \otimes f_0^n, \phi \rangle - \langle \delta_{t=0} \otimes f_0, \phi \rangle| \le \int_{\mathbb{R}^N} |f_0^n - f_0| \phi(0,x) dx \le sup_{x\in\mathbb{R}^N} |\phi(0,x)| \|f_0^n - f_0\|_1$
hence the sequence $\delta_{t=0} \otimes f_0^n$ converges to $\delta_{t=0} \otimes f_0$ in $\mathcal{D}'(\mathbb{R}_t \times \mathbb{R}^N)$,which implies the convergence of the sequence $(E_N * (\delta_{t=0} \otimes f_0^n)$ to $E_N * (\delta_{t=0} \otimes f_0)$ in $\mathcal{D}'(\mathbb{R}_t \times \mathbb{R}^N)$.Then this convergence is also valid $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{R}^N)$. So we can conclude that when t>0 and $f_0 \in L^1(\mathbb{R}^N)$:

$$E_N * (\delta_{t=0} \otimes f_0) = E_N(t,.) * f_0(.)$$

-$E_N * (\delta_{t=0} \otimes f_0) \in L^1(\mathbb{R}^N)$ for t>0 so $E_N * \left(\delta_{t=0} \otimes f_0 + \dot{S}\right) \in L^1(\mathbb{R}^N)$ for t>0

hence $\mathcal{F}((-\Delta)^{\frac{\beta}{2}}(E_N * \left(\delta_{t=0} \otimes f_0 + \dot{S}\right))) = |\xi|^\beta \mathcal{F}((E_N * \left(\delta_{t=0} \otimes f_0 + \dot{S}\right) =$

$$|\xi|^{\beta}\mathcal{F}(E_N)\mathcal{F}\left(\delta_{t=0}\otimes f_0+\dot{S}\right)=\mathcal{F}((-\Delta)^{\frac{\beta}{2}}E_N)\mathcal{F}\left(\delta_{t=0}\otimes f_0+\dot{S}\right)=$$

$$\mathcal{F}((-\Delta)^{\frac{\beta}{2}}E_N)\mathcal{F}\left(\delta_{t=0}\otimes f_0+\dot{S}\right)=\mathcal{F}((-\Delta)^{\frac{\beta}{2}}E_N)*\left(\delta_{t=0}\otimes f_0+\dot{S}\right))$$ ($\mathcal{F}$ the symbol of Fourier transform),

hence $(-\Delta)^{\frac{\beta}{2}}(E_N*\left(\delta_{t=0}\otimes f_0+\dot{S}\right))=((-\Delta)^{\frac{\beta}{2}}E_N)*\left(\delta_{t=0}\otimes f_0+\dot{S}\right)$

So we have :

$$(\partial_t+(-\Delta_x)^{\frac{\beta}{2}})(E_N*(\delta_{t=0}\otimes f_0+\dot{S})=((\partial_t+(-\Delta_x)^{\frac{\beta}{2}})E_N)*(\delta_{t=0}\otimes f_0+\dot{S}) \text{ in } \mathcal{D}'(\mathbb{R}_t\times\mathbb{R}_x^N)$$

while

$$\operatorname{supp}(E_N*(\delta_{t=0}\otimes f_0+\dot{S}))\subset\operatorname{supp}(E_N)+\operatorname{supp}(\delta_{t=0}\otimes f_0+\dot{S})\subset(\mathbb{R}_+\times\mathbb{R}^N)+(\mathbb{R}_+\times\mathbb{R}^N)$$
$$\subset\mathbb{R}_+\times\mathbb{R}^N$$

Let us pass to the uniqueness of the solution in the sense of the tempered distributions f of the Cauchy problem for the heat equation. Suppose there are another, say g, and set $h=f-g$

Then the distribution $h\in S'(\mathbb{R}_t\otimes\mathbb{R}_x^N)$ and check the conditions:

$$\partial_t h+(-\Delta)^{\frac{\beta}{2}}h=0 \quad ,t>0$$

Applying Lemma 4, we find that $h=f-g$, hence the uniqueness announced.

3.3 Case where the initial data in L^2

Proposition 5

Let $N\geq 1$ be ,and f_0 is an initial data in $L^1(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$. There is then a unique solution within the meaning of the tempered distributions of the Cauchy problem:

$$\partial_t f+(-\Delta_x)^{\frac{\beta}{2}}f=0 \qquad x\in\mathbb{R}^N, t>0$$
$$f_{|t=0}=f_0$$

The restriction of this solution f to $\mathbb{R}_+^*\times\mathbb{R}^N$ (for t = 0) into a function belonging to $C(\mathbb{R}_+,L^2(\mathbb{R}^N))$ and given by the formula :

$$f(t,x)=E(t,.)*f_0(x)=\int_{\mathbb{R}^N}E(t,x-y)f_0(y)dy \quad a.e \;,\; x\in\mathbb{R}^N, t>0$$

and $$f(0,x)=f_0(x) \quad a.e\,,\; x\in\mathbb{R}^N$$

Proof :

For all $n\geq 1,$, define f_n as the solution in the sense of tempered distributions of the Cauchy problem for the heat equation without second member and with initial end data f_0^n defined by

$$f_0^n(x)=1_{B(0,n)}(x)f_0(x) \quad x\in\mathbb{R}^N$$

Obviously, $f_0^n\in\mathcal{E}'(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$ so that $f^n=E_N*(\delta_{t=0}\otimes f_0^n)$ (see proposition 4)

Apply the partial Fourier transform at x to each member of equality above. Noting the dual Fourier variable of x and $\hat{f}$ the transform of Fourier of partial f into the variable x, we find that

$$\hat{f}^n(t,\xi)=\hat{E}(t,\xi)\,\hat{f}_0^n(\xi)=e^{-t|\xi|^{\beta}}\,\hat{f}_0^n(\xi) \text{ a.e on } \xi\in\mathbb{R}^N \text{ for all t>0}$$

Let, for all t> 0, be the measurable function $\xi \to g(t,\xi)$ defined by $g(t,\xi) = e^{-t|\xi|^\beta} \hat{f}_0(\xi)$
we obviously have $g(t,.) \in L^2(\mathbb{R}^N)$ with $\|g(t,.)\|_{L^2(\mathbb{R}^N)} \le \|f_0\|_{L(\mathbb{R}^N)}$

Since $0 \le e^{-t|\xi|^\beta} \le 1$ for all $\xi \in \mathbb{R}^N$ and all $t \ge 0$. By the theorem of Plancherel, there exists, for all $t \ge 0$, a unique function $x \to f(t,x)$ belonging to $L^2(\mathbb{R}^N)$ and such that :

$$\hat{f}(t,\xi) = g(t,\xi) = e^{-t|\xi|^\beta} \hat{f}_0(\xi) \quad \text{a.e on } \xi \in \mathbb{R}^N \text{, for all t>0}$$

Let us also denote by f^n and f the extensions of f^n and f by 0 for t <0.Always thanks to Plancherel's theorem, for all $t \ge 0$, we have

$$\|f^n(t,.) - f(t,.)\|_{L^2(\mathbb{R}^N)} = \frac{1}{(2\pi)^N} \left\| \hat{f}^n(t,.) - \hat{f}(t,.) \right\|_{L^2(\mathbb{R}^N)} \le \frac{1}{(2\pi)^N} \left\| \hat{f}_0^n - \hat{f}_0 \right\|$$
$$= \|f_0^n - f_0\|_{L^2(\mathbb{R}^N)} \to 0$$

when $n \to +\infty$, and in particular, by dominated convergence

$$f^n \to f \ \text{ in } \mathcal{D}'(\mathbb{R}_t \times \mathbb{R}_x^N) \text{ for } n \to +\infty$$

So $$\partial_t f^n \to \partial_t f \text{ and } (-\Delta)^{\frac{\beta}{2}} f^n \to (-\Delta)^{\frac{\beta}{2}} f \text{ in } \mathcal{D}'(\mathbb{R}_t \times \mathbb{R}_x^N)$$

By the proposition 4, $\partial_t f^n + (-\Delta)^{\frac{\beta}{2}} f^n = \delta_{t=0} \otimes f_0^n$

and $\delta_{t=0} \otimes f_0^n \to \delta_{t=0} \otimes f_0$ in $\mathcal{D}'((\mathbb{R}_t \times \mathbb{R}_x^N)$

we deduce that $\partial_t f + (-\Delta)^{\frac{\beta}{2}} f = \delta_{t=0} \otimes f_0$ in $\mathcal{D}'(\mathbb{R}_t \times \mathbb{R}_x^N)$
On the other hand, by construction $\operatorname{supp}(f^n) \subset \mathbb{R}_+ \times \mathbb{R}^N$ so that $\operatorname{supp}(f) \subset \mathbb{R}_+ \times \mathbb{R}^N$
The uniqueness of this solution is obtained as in proposition 4, by a direct application of Lemma 4.

The formula $\hat{f}(t,\xi) = e^{-t|\xi|^\beta} \hat{f}_0(\xi)$ a.e on $\xi \in \mathbb{R}^N$ for all t>0 and Plancherel's theorem show that

$$f_{|\mathbb{R}_+ \times \mathbb{R}^N} \in C(\mathbb{R}_+; L^2(\mathbb{R}^N))$$

Indeed, for all $t \ge 0$ and any sequence $t_n \ge 0$ such that $t_n \to t$, we have

$$\|f(t_n,.) - f(t,.)\|_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (e^{-t_n|\xi|^\beta} - e^{-\partial|\xi|^\beta})^2 \left|\hat{f}_0(\xi)\right|^2 d\xi \to 0$$

when $n \to \infty$ by dominated convergence, since

$$\left|e^{-t_n|\xi|^\beta} - e^{-t|\xi|^\beta}\right|^2 \left|\hat{f}_0(\xi)\right|^2 \le 2\left|\hat{f}_0(\xi)\right|^2 \quad \text{a.e on } \xi \in \mathbb{R}^N$$

Because $\left|e^{-t_n|\xi|^\beta} - e^{-t|\xi|^\beta}\right|^2 \le e^{-2t_n|\xi|^\beta} + e^{-t|\xi|^\beta} \le 2$

Corollary 1

Let $N \ge 1$be, an initial data $f_0 \in L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and $S \in C(\mathbb{R}_+, L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))$ such as

$$\sup_{t \ge 0} \int_{\mathbb{R}^N} |S(t,x)|^2 dx < \infty$$

There then exists a unique $f \in C(\mathbb{R}_+, L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))$ whose extension by 0 for t <0 is a solution in the sense of the tempered distributions of the Cauchy problem

$$\partial_t f + (-\Delta)^{\frac{\beta}{2}} f = S \quad x \in \mathbb{R}^N,\ t>0$$
$$f_{|t=0} = f_0$$

This solution is given by the formula (see Proposition 2)

$$f(t,.) = P(t) f_0 + \int_0^t P(t-s)S(s,.)ds \quad \text{for all } t>0$$

*For $P(t)$ see propriety 2

The above formula giving f is called the "Duhamel formula ".She is still written equivalently in the form

$$f(t,x) = (E_N(t,.) * f_0)(x) + E_N *_{i,x} (1_{\mathbb{R}_+}(t)S)(t,x)$$

Proof:

For any integer $n \geq 1$, note:

$S_n(t,x) = 1_{\left[\frac{1}{n},n\right]}(t) 1_{\overline{B(0,n)}}(x) S(t,x)$ a.e in $x \in \mathbb{R}^N$ and for all $t \geq 0$

Obviously $S_n \in \mathcal{E}'(\mathbb{R}_+^* \times \mathbb{R}^N)$ and $S_n(t,.) \in L^1(\mathbb{R}^N)$. Let $g_n = E_N * \dot{S}_n$ where $\dot{S}_n$ denotes the extension of S_n by 0 for $t \leq 0$. Note that g_n is written

$$g_n(t,.) = \int_0^t P(t-s) S_n(s,.)ds \quad \text{if } t<0$$

By proposition 4, g_n is a solution in the sense of tempered distributions of the Cauchy problem

$$\partial_t g_n + (-\Delta)^{\frac{\beta}{2}} g_n = S_n \quad x \in \mathbb{R}^N,\ t>0$$
$$g_{n|t=0} = 0$$

On the other hand, for all t> 0 and all $n \geq t$, we have

$$\left\| \int_0^t P(t-s) S_n(s,.)ds - \int_0^t P(t-s)S(s,.)ds \right\|_{L^2(\mathbb{R}^N)} \leq \int_0^t \left\| P(t-s)S_n(s,.) - P(t-s)S(s,.) \right\|_{L^2(\mathbb{R}^N)} ds$$

$$\leq \int_0^{\frac{1}{n}} \|S(s,.)\|_{L^2(\mathbb{R}^N)} ds + \int_{\frac{1}{n}}^t \|S_n(s,.) - S(s,.)\|_{L^2(\mathbb{R}^N)} ds$$

see (2.8) and the definition of S_n

For the first term on the right hand side, we have

$$\int_0^{\frac{1}{n}} \|S(s,.)\|_{L^2(\mathbb{R}^N)} ds \leq \frac{1}{n} \sup \|S(s,.)\|_{L^2(\mathbb{R}^N)} \to 0 \text{ when } n \to \infty$$

For the second term, we start by applying to the integral at t the equality of Cauchy-Schwarz, in we observe, thanks to the definition of S_n , that, for all T> 0 and all $t \in [0,T]$

$$\int_{\frac{1}{n}}^t \|S_n(s,.) - S(s,.)\|_{L^2(\mathbb{R}^N)} ds \leq \sqrt{T} \left(\int_0^T \int_{\mathbb{R}^N} |S(s,x)|^2 1_{\mathbb{R}^N \setminus \overline{B(0,n)}} dxds \right)^{\frac{1}{2}} \to 0$$

when $n \to \infty$ by dominated convergence. Posing

$$g(t,.)=\int_0^t P(t-s)S(s,.)ds \text{ if } t\geq 0,\ \ g(t,.)=0 \text{ if t<0}$$

Hence we have shown that $g_n(t,.)\to g(t,.)$ in $L^2(\mathbb{R}^N)$ uniformly on $t\in[0,T]$, for all T> 0.

Note that $g(t,.)\in L^1(\mathbb{R}^N)$ (see 2.8)

In particular, $g_n \to g$ in $\mathcal{D}'(\mathbb{R}\times\mathbb{R}^N)$, and passing to the limit in the sense of distributions in the equality $\partial_t g_n+(-\Delta)^{\frac{\beta}{2}} g_n=\dot{S}_n$ in $\mathcal{S}'((\mathbb{R}\times\mathbb{R}^N)$

we find that g is a solution in the sense of the tempered distributions of the Cauchy problem

$$\partial_t g+(-\Delta)^{\frac{\beta}{2}} g=S \quad x\in\mathbb{R}^N, t>0$$
$$g_{|t=0}=0$$

Check that the restriction of g to $\mathbb{R}_+\times\mathbb{R}^N$ define an element of $C(\mathbb{R}_+, L^2(\mathbb{R}^N)\cap L^1(\mathbb{R}^N))$ when $L^1(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$ has the norm in $L^2(\mathbb{R}^N)$ .Indeed, for $t,t'\in[0,T]$ with $t\leq t'$, we have, using the property 2

$$\left\|g(t',.)-g(t,.)\right\|_{L^2(\mathbb{R}^N)}=\left\|\int_0^{t'} P(t'-s)S(s,.)ds-\int_0^t P(t-s)S(s,.)ds\right\|_{L^2(\mathbb{R}^N)}$$

$$\leq\left\|\int_0^t P(t-s)(P(t'-t)-I)S(s,.)ds\right\|_{L^2(\mathbb{R}^N)}+\left\|\int_t^{t'} P(t'-s)S(s,.)ds\right\|_{L^2(\mathbb{R}^N)}$$

$$\leq\int_0^t\left\|(P(t'-t)-I)S(s,.)\right\|_{L^2(\mathbb{R}^N)}ds+\int_t^{t'}\left\|P(t'-s)S(s,.)\right\|_{L^2(\mathbb{R}^N)}ds$$

$$\leq\int_0^T\left\|(P(t'-t)-I)S(s,.)\right\|_{L^2(\mathbb{R}^N)}ds+\int_t^{t'}\left\|S(s,.)\right\|_{L^2(\mathbb{R}^N)}ds$$

The second term on the right hand side verifies

$\int_t^{t'}\left\|S(s,.)\right\|_{L^2(\mathbb{R}^N)}ds\leq(t'-t)\sup_{s\in[0,T]}\left\|S(s,.)\right\|_{L^2(\mathbb{R}^N)}\to 0$ when $t'-t\to 0$. As for the first term, it converges to 0 by convergence dominated when $t'-t\to 0$ since

$$\left\|(P(t'-t)-I)S(s,.)\right\|_{L^2(\mathbb{R}^N)}\to 0$$

according to the propriety 2 and that ,. $\left\|(P(t'-t)-I)S(s,.)\right\|_{L^2(\mathbb{R}^N)}\leq 2\sup_{s\in[0,T]}\left\|S(s,.)\right\|_{L^2(\mathbb{R}^N)}$

for all $s\in[0,T]$ thanks to this same propriety. We conclude that $g_{|\mathbb{R}_+\times\mathbb{R}^N}\in C(\mathbb{R}_+;L^2(\mathbb{R}^N)\cap L^1(\mathbb{R}^N))$ when $L^1(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$ has the norm in $L^2(\mathbb{R}^N)$.

Define $f_1(t,.)=P(t)f_0$ when $t\geq 0$, $f_1(t,.)=0$ when t<0

Hence we can say that f_1 is a solution in the sense of tempered distributions of the Cauchy problem

$$\partial_t f_1+(-\Delta)^{\frac{\beta}{2}} f_1=0,\ x\in\mathbb{R}^N, t>0$$
$$f_{1|t=0}=f_0$$

and $f_{1|\mathbb{R}_+\times\mathbb{R}^N}\in C(\mathbb{R}_+, L^1(\mathbb{R}^N)\cap L^2(\mathbb{R}^N))$

As the heat equation is linear, we then deduce from the above that $f = f_1 + g$ is a solution in the sense of the tempered distributions of the problem of Cauchy

$$\partial_t f + (-\Delta)^{\frac{\beta}{2}} f = S \qquad x \in \mathbb{R}^N, t > 0$$
$$f_{|t=0} = f_0$$

It is the only one, because if there was another one, the unicity deduce from the lemma 4 ,

And we can say that $f_{/\mathbb{R}_+ \times \mathbb{R}^N} \in C(\mathbb{R}_+, L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N))$ because $f_{1/\mathbb{R}_+ \times \mathbb{R}^N} \in C(\mathbb{R}_+, L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N))$ and $g_{|\mathbb{R}_+ \times \mathbb{R}^N} \in C(\mathbb{R}_+; L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))$

Conclusion

The proofs of all the properties described in this Cauchy problem for heat equation with fractional Laplacian are inspired by the proofs of the properties of the Cauchy problem of heat equation

$$\partial_t f - \frac{1}{2}\Delta_x = S, x \in \mathbb{R}^N, t > 0$$
$$f_{|t=0} = f_0$$

The question is:
Can we do the same method with the Schrodinger equation?

Bibliograhie

[1] F.Golse .*Distributions, Analyse de Fourier, Equations aux Dérivées partielles* Octobre 2012

[2] H.Boumaza,''''*Théorie des distributions''*.UPN-Sup Galilée-Formation des ingénieurs MACS.Année scolaire 2015-2016

[3] Ionel Sorin Ciuperca *Cours et méthodes mathémateiques pour les ingénieurs.* Septembre 2015

[4] G.Abramovici.*Théorie des distributions*.Université Paris .Octobre 2013

[5] V.S Vladmirov.*Generalized functions in mathematical physics.* Mir Publishers Moscow 1979

[6] Laurent Schwartz*,'' Théorie des distributions.''* Publications de l'Institut de Mathématique de l'Université de Strasbourg. Hermann. Paris 1966.

[7] A.S. Demidov. *Generalized functions in mathematical physics. Main ideas and concepts*

[8] A Fino, M Kirane. *The Cauchy problem for heat equation with fractional Laplacian and exponential nonlinearity*

[9] Thomas Cluzeau. *Mathématiques pour l'Ingénieur* École Nationale Supérieure d'Ingénieurs de Limoges 16 rue d'atlantis, Parc ester technopole 87068 Limoges CEDEX

[10] Pablo Raúl Stinga. *User's guide to the fractional Laplacian and the method of semigroups* Department of Mathematics, Iowa State University, 396 Carver Hall, Ames, IA 50011, United States of America, 15Aug 2018

[11] Celso Martinez Carracedo and Miguel Sanz. *The Theory of Fractional Powers of Operators Alix*. Departament de Matematica Aplicada Universitat de Valencia C/Dr. Moliner 50 46100 Burjassot Valencia Spain.

[12] Mateusz Kwasnicki. *Ten Equivalent Definitions of the Fractional Laplace Opérator* 14 sept 2015

[13] O. Ciaurri, L. Roncal, P. R. Stinga, J. L. Torrea and J. L. Varona, Nonlocal discrete diffusion equations and the fractional discrete Laplacian,regularity and applications, *Adv. Math.* 330 (2018), 688–738

[14] L. Silvestre, PhD thesis, *The University of Texas at Austin*, 2005.

[15] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, *Comm. Pure Appl. Math.* **60** (2007), 67–112.